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On estimation of the Hurst index of solutions of stochastic integral equations*

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Abstract. Let X be a solution of a stochastic integral equation driven by a fractional Brownian motion B^H and let $V_n(X,2)=\sum_{k=1}^{n}(\Delta_k X)^2$, where $\Delta_k X = X(\frac{k+1}{n}) - X(\frac{k}{n})$. We study the question under what conditions $n^{2H-1}V_n(X,2)$ convergence almost surely as $n\to\infty$ holds. This fact is used to obtain a strongly consistent estimator of the Hurst index H, $1/2 < H < 1$.

Keywords: fractional Brownian motion, quadratic variation, consistent estimator.

1. Introduction

A process is often observed at discrete points of time. It is natural to draw an inference about various properties of a process from these observations. For example, in finance a quadratic variation may be used to determine an integrated power volatility.

In this paper, we consider the stochastic integral equation

$$X(t) = \xi + \int_0^t g(X(s))\,\mathrm{d}B_s^H, \quad t\in[0,1], \tag{1}$$

where B^H is a fractional Brownian motion (fBm) with the Hurst index $1/2 < H < 1$. The integral is Riemann-Stieltjes defined pathwise. It is known (see, e.g., [4], Section 5.3) that almost all sample paths of fBm B^H, $0 < H < 1$, have locally bounded p-variation for $p > 1/H$.

For $0 < \alpha \leqslant 1$, $C^{1+\alpha}(\mathbb{R})$ denotes the set of all C^1-functions $g\colon \mathbb{R}\to\mathbb{R}$ such that

$$\|g\|_\alpha := |g|_\infty + |g|_\alpha = \sup_x |g'(x)| + \sup_{x\neq y}\frac{|g'(x)-g'(y)|}{|x-y|^\alpha} < \infty.$$

Let $g\in C^{1+\alpha}(\mathbb{R})$, $0<\alpha\leqslant 1$. For $1\leqslant p < 1+\alpha$ there exists a unique solution of equation (1) with almost all sample paths in the class of all continuous functions defined on $[0,1]$ with bounded p-variation (see [2], [5]).

For a real-valued process $Y=(Y_t)$, $t\in[0,1]$, we define the quadratic variation as $V_n(Y,2)=\sum_{k=1}^{n}(\Delta_k Y)^2$, where $\Delta_k^n Y = Y(t_k^n) - Y(t_{k-1}^n)$, $t_k^n = \frac{k}{n}$. For simplicity, we

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omit index n for t in the sequel. The asymptotic behavior of a second order quadratic variations of Gaussian processes was considered in [1] (see also references in [1]).

The main result of this paper is the following theorem.

THEOREM 1. *Let* $g \in C^{1+\alpha}(\mathbb{R})$, $0 < \alpha < 1$. *Then*

$$n^{2H-1} V_n(X,2) \xrightarrow{\mathbf{a}.s.} \int_0^1 g^2(X(t))\,\mathrm{d}t.$$

It is easy to see that the following corollary holds.

COROLLARY 2. *Define*

$$\widehat{H}_n := \frac{1}{2} - \frac{1}{2\ln 2} \ln \frac{V_{2n}(X,2)}{V_n(X,2)}.$$

Then $\widehat{H}_n - H \xrightarrow{\mathbf{a}.s.} 0$ *as* $n \to \infty$.

2. Basic notions and auxiliary results

Let

$$\mathcal{W}_p([a,b]) := \bigl\{f\colon [a,b] \to \mathbb{R}\colon\ v_p(f;[a,b]) < \infty\bigr\},$$

where

$$v_p\bigl(f;[a,b]\bigr) = \sup_{\varkappa} \sum_{k=1}^{n} \bigl|f(x_k) - f(x_{k-1})\bigr|^p,$$

$\varkappa = \{x_i\colon i = 0,\dots,n\}$ being all finite partitions of $[a,b]$ such that $a = x_0 < x_1 < \dots < x_n = b$.

Let $V_p(f) := V_p(f;[a,b]) = v_p^{1/p}(f;[a,b])$. $V_p(f)$ is a non-increasing function of p, that is, if $0 < q < p$, then $V_p(f) \leqslant V_q(f)$.

We used here the following results. Firstly, we obtain from [9] that

$$\left|\int_a^b f\,\mathrm{d}h - f(y)\bigl[h(b) - h(a)\bigr]\right| \leqslant C_{p,q} V_q\bigl(f;[a,b]\bigr) V_p\bigl(h;[a,b]\bigr) \tag{2}$$

holds for any $y \in [a,b]$, where $C_{p,q} = \zeta(p^{-1} + q^{-1})$ and $\zeta(s) = \sum_{n \geqslant 1} n^{-s}$.

Let $f \in \mathcal{W}_q([a,b])$ and $h \in \mathcal{W}_p([a,b])$. It follows from (2) that

$$V_p\left(\int_a^{\cdot} f\,\mathrm{d}h;[a,b]\right) \leqslant C_{p,q} V_{q,\infty}\bigl(f;[a,b]\bigr) V_p\bigl(h;[a,b]\bigr), \tag{3}$$

where $V_{q,\infty}(f;[a,b]) = V_q(f;[a,b]) + |f|_{\infty,[a,b]}$, $|f|_{\infty,[a,b]} = \sup_{a \leqslant x \leqslant b} |f(x)|$.

Secondly, we have Young's version of Hölder's inequality

$$\sum_{k=1}^{n} |a_k b_k| \leqslant \Bigl(\sum_{k=1}^{n} |a_k|^p\Bigr)^{1/p} \Bigl(\sum_{k=1}^{n} |b_k|^q\Bigr)^{1/q}, \tag{4}$$

that is valid for any $p, q > 0$ such that $1/p + 1/q \geqslant 1$.

Let $a < c < b$ and $f \in \mathcal{W}_p([a, b])$ with $0 < p < \infty$. It is known (e.g., [2]) that

$$V_{p,\infty}(fg; [a, b]) \leqslant V_{p,\infty}\big(f; [a, b]\big) V_{p,\infty}\big(g; [a, b]\big). \tag{5}$$

Also, let $f \in \mathcal{W}_q([a, b])$ and $g \in \mathcal{W}_p([a, b])$. For any partition $\varkappa$ and for $p^{-1} + q^{-1} \geqslant 1$, we have by Hölder inequality (see, e.g., [9])

$$\sum_i V_q\big(f; [x_{i-1}, x_i]\big)\, V_p\big(g; [x_{i-1}, x_i]\big) \leqslant V_q\big(f; [a, b]\big)\, V_p\big(g; [a, b]\big). \tag{6}$$

Since almost all the sample paths of B^H, $1/2 \leqslant H < 1$, are locally Hölder continuous (see, e.g., [8]), we have

$$V_p\big(B^H; [s, t]\big) \leqslant L_T^{H,1/p}\, (t - s)^{1/p}, \tag{7}$$

where $s < t \leqslant T,\ p > 1/H,$

$$L_T^{H,\gamma} = \sup_{\substack{s \neq t \\ s,t \leqslant T}} \frac{|B_t^H - B_s^H|}{|t - s|^{\gamma}}, \quad 0 < \gamma < H, \quad \mathbf{E}\big(L_T^{H,\gamma}\big)^k < \infty \ \ \forall k \geqslant 1.$$

3. Proofs

By applying the generalized Itô formula (see [6]) to $(\Delta_k X)^2$ we get

$$\begin{aligned}
(\Delta_k X)^2 &= \left(\int_{t_{k-1}}^{t_k} g(X(s))\, \mathrm{d}B_s^H\right)^2 \\
&= 2\int_{t_{k-1}}^{t_k} \left(\int_{t_{k-1}}^{s} \big[g(X(u)) - g(X(t_{k-1}))\big]\, \mathrm{d}B_u^H\right) g(X(s))\, \mathrm{d}B_s^H \\
&\quad +2\int_{t_{k-1}}^{t_k} \left(\int_{t_{k-1}}^{s} g(X(t_{k-1}))\, \mathrm{d}B_u^H\right)\big[g(X(s)) - g(X(t_{k-1}))\big]\, \mathrm{d}B_s^H \\
&\quad +2\big(g(X(t_{k-1}))\big)^2 \int_{t_{k-1}}^{t_k} \left(\int_{t_{k-1}}^{s} \mathrm{d}B_u^H\right) \mathrm{d}B_s^H.
\end{aligned}$$

We will show that the first two terms of the sum converge to 0 as $n \to \infty$. From (2) and (5) we obtain

$$\begin{aligned}
&\left|\int_{t_{k-1}}^{t_k} \left(\int_{t_{k-1}}^{s} \big[g(X(u)) - g(X(t_{k-1}))\big]\, \mathrm{d}B_u^H\right) g(X(s))\, \mathrm{d}B_s^H\right| \\
&\quad \leqslant C_{p,q} V_{q,\infty}\left(\int_{t_{k-1}}^{\cdot} \big[g(X(u)) - g(X(t_{k-1}))\big]\, \mathrm{d}B_u^H \cdot g(X); [t_{k-1}, t_k]\right) \\
&\qquad \times V_p\big(B^H; [t_{k-1}, t_k]\big)
\end{aligned}$$

$$\leqslant C_{p,q} V_{q,\infty}\biggl(\int_{t_{k-1}}^{\cdot}\bigl[g(X(u)) - g(X(t_{k-1}))\bigr]\,\mathrm{d}B_u^H; [t_{k-1}, t_k]\biggr)$$
$$\times V_{q,\infty}\bigl(g(X); [t_{k-1}, t_k]\bigr) V_p\bigl(B^H; [t_{k-1}, t_k]\bigr).$$

It follows from (2) and (3) that

$$V_{q,\infty}\biggl(\int_{t_{k-1}}^{\cdot}\bigl[g(X(u)) - g(X(t_{k-1}))\bigr]\,\mathrm{d}B_u^H; [t_{k-1}, t_k]\biggr)$$
$$\leqslant 3C_{p,q} V_q\bigl(g(X); [t_{k-1}, t_k]\bigr) V_p\bigl(B^H; [t_{k-1}, t_k]\bigr).$$

Consequently, from (4), (6), and (7) we get

$$\sum_{k=1}^{n}\biggl|\int_{t_{k-1}}^{t_k}\biggl(\int_{t_{k-1}}^{s}\bigl[g(X(u)) - g(X(t_{k-1}))\bigr]\,\mathrm{d}B_u^H\biggr) g(X(s))\,\mathrm{d}B_s^H\biggr|$$
$$\leqslant 3C_{p,q}^2 \sum_{k=1}^{n} V_q\bigl(g(X); [t_{k-1}, t_k]\bigr) V_{q,\infty}\bigl(g(X); [t_{k-1}, t_k]\bigr) V_p^2\bigl(B^H; [t_{k-1}, t_k]\bigr)$$
$$\leqslant 6C_{p,q}^2 \max_{1\leqslant k\leqslant n}\bigl[V_p\bigl(B^H; [t_{k-1}, t_k]\bigr)\bigr] V_q^2\bigl(g(X); [0,1]\bigr) V_p\bigl(B^H; [0,1]\bigr)$$
$$\leqslant 6n^{-1/p} C_{p,q}^2 L_1^{H,1/p} V_q^2\bigl(g(X); [0,1]\bigr) V_p\bigl(B^H; [0,1]\bigr).$$

Similarly, for the second term of the sum we have

$$\biggl|\int_{t_{k-1}}^{t_k}\biggl(\int_{t_{k-1}}^{s} g(X(t_{k-1}))\,\mathrm{d}B_u^H\biggr)\bigl[g(X(s)) - g(X(t_{k-1}))\bigr]\,\mathrm{d}B_s^H\biggr|$$
$$\leqslant 4C_{p,q} |g|_\infty V_q\bigl(g(X)); [t_{k-1}, t_k]\bigr) V_p^2\bigl(B^H; [t_{k-1}, t_k]\bigr),$$

and

$$\sum_{k=1}^{n}\biggl|\int_{t_{k-1}}^{t_k}\biggl(\int_{t_{k-1}}^{s} g(X(t_{k-1}))\,\mathrm{d}B_u^H\biggr)\bigl[g(X(s)) - g(X(t_{k-1}))\bigr]\,\mathrm{d}B_s^H\biggr|$$
$$\leqslant 4n^{-1/p} C_{p,q} |g|_\infty L_1^{H,1/p} V_q\bigl(g(X); [0,1]\bigr) V_p\bigl(B^H; [0,1]\bigr).$$

Note that $V_q(g(X); [0,1]) \leqslant |g'|_\infty V_q(X; [0,1])$ and

$$V_q\bigl(X; [0,1]\bigr) \leqslant \frac{C_{p,q/\alpha}|g|_\infty}{1-\alpha} V_p\bigl(B^H; [0,1]\bigr) + \bigl[C_{p,q/\alpha}|g|_\alpha V_p\bigl(B^H; [0,1]\bigr)\bigr]^{1/(1-\alpha)}.$$

Therefore

$$n^{2H-1}\Biggl[\sum_{k=1}^{n}\biggl|\int_{t_{k-1}}^{t_k}\biggl(\int_{t_{k-1}}^{s}\bigl[g(X(u)) - g(X(t_{k-1}))\bigr]\,\mathrm{d}B_u^H\biggr) g(X(s))\,\mathrm{d}B_s^H\biggr|$$

$$+\sum_{k=1}^{n}\left|\int_{t_{k-1}}^{t_k}\left(\int_{t_{k-1}}^{s} g(X(t_{k-1}))\,\mathrm{d}B_u^H\right)\left[g(X(s))-g(X(t_{k-1}))\right]\mathrm{d}B_s^H\right|\Bigg] \xrightarrow{\mathbf{a}.s.} 0,$$

as $n \to \infty$. To this end it suffices to take p sufficiently close to H and such that $2H < 1 + 1/p$.

Consequently, the theorem will be proved if

$$2n^{2H-1}\sum_{k=1}^{n}\left[\left(g(X(t_{k-1}))\right)^2\int_{t_{k-1}}^{t_k}\left(\int_{t_{k-1}}^{s}\mathrm{d}B_u^H\right)\mathrm{d}B_s^H\right] \xrightarrow{\mathbf{a}.s.} \int_0^1 g^2(X(t))\,\mathrm{d}t\,.$$

By the generalized Itô formula we get

$$2\left(g(X(t_{k-1}))\right)^2\int_{t_{k-1}}^{t_k}\left(\int_{t_{k-1}}^{s}\mathrm{d}B_u^H\right)\mathrm{d}B_s^H = \left(g(X(t_{k-1}))\right)^2(\Delta_k B^H)^2.$$

Put

$$S_n(t) = n^{2H-1}\sum_{k=1}^{[nt]}\left(\Delta_k B^H\right)^2, \quad t \in [0,1].$$

Then

$$2n^{2H-1}\sum_{k=1}^{n}\left[\left(g(X(t_{k-1}))\right)^2\int_{t_{k-1}}^{t_k}\left(\int_{t_{k-1}}^{s}\mathrm{d}B_u^H\right)\mathrm{d}B_s^H\right] = \int_0^1 g^2(X(t))\,\mathrm{d}S_n(t).$$

It is known that (see [4])

$$S_n(t) \xrightarrow{\mathbf{a}.s.} t.$$

Since the function $S_n(t)$ is non-decreasing, we have (see Lemma 1 in [7])

$$\sup_{t\leqslant 1}|S_n(t)-t| \xrightarrow{\mathbf{a}.s.} 0 \quad \text{as } n\to\infty\,.$$

So $S_n(t)$ is uniformly bounded. Consequently, the Helly-Bray theorem implies that

$$\int_0^1 g^2(X(t))\,\mathrm{d}S_n(t) \xrightarrow{\mathbf{a}.s.} \int_0^1 g^2(X(t))\,\mathrm{d}t \quad \text{as } n\to\infty\,.$$

This completes the proof of the theorem.

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REZIUMĖ

K. Kubilius, D. Melichov. Stochastinių integralinių lygčių sprendinių Hursto indekso įvertinio klausimu

Nagrinėjamas stochastinių integralinių lygčių sprendinių kvadratinės variacijos asimptotinis elgesys, kuris leidžia gauti stipriai pagrįstą Hursto indekso H, $1/2 < H < 1$, įvertinį.

Raktiniai žodžiai: trupmeninis Brauno judesys, kvadratinė variacija, pagrįstas įvertinys.