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H. Bergstrom's asymptotic expansions

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Abstract. In this work it is researched H. Bergstrom work about asymptotic behavior of convolutions of probability distributions. We give the theorem about the members of expansion in decreasing order and the remainder term estimation. There we propose apply Levy–Scheffer polynomials and to construct pseudo-moments for more profound H. Bergstrom's asymptotic expansion investigations.

Keywords: Bergstrom identity, Levy-Scheffer polynomials.

In 1951 H. Bergstrom [1] published equation about two probability distributions F and G compositions F^{*n} and G^{*n} difference:

$$F^{*n}(B) - G^{*n}(B) = \sum_{\nu=1}^{s} C_n^{\nu} G^{*(n-\nu)} * (F-G)^{*\nu}(B) + r_n^{(s+1)}(B), \tag{1}$$

where $s \leqslant n$,

$$r_n^{(s+1)}(B) = \sum_{\mu=s+1}^{n} C_{\mu-1}^{s} F^{*(n-\mu)} * (F-G)^{*(s+1)} * G^{*(\mu-s-1)}(B). \tag{2}$$

F and G probability distributions in k-dimensional Euclidean space $\mathbb{R}^k$, and $B \in \mathrm{B}(\mathbb{R}^k)$ – Borel σ-algebra.

Using compositions method H. Bergstrom proved few theorems. Let's mark:

$$V\big[f(\vec{x})\big] \text{ – functions } f(\vec{x}),\ \ \vec{x} \in \mathbb{R}^k,\ \text{complete variation,} \tag{3}$$

$$M_A[f] = \max_{\vec{x} \in \mathbb{R}^k} \big|f(A-\vec{x})\big|,\ \text{ where } A-\vec{x} = \big\{\vec{y}-\vec{x}\colon\ \vec{y} \in A,\ \vec{x} \in \mathbb{R}^k\big\}. \tag{4}$$

Let's say $g(u)$, $u \in \mathbb{R}$ – non-negative function, which has characteristic:

$$g\left(\frac{u}{\rho}\right) = \mathrm{O}\big(g(u)\big), \quad \text{when } \rho \text{ is positive constant aand } u \to \infty, \text{ i.e.,} \tag{5}$$

$$g\left(\frac{u}{\rho}\right) < Cg(u), \quad \text{where } C \text{ independent from } u,\ \text{positive constant.}$$

Lets indicate:

$$\Delta_n^{(\nu)}(A) = C_n^{\nu} G^{*(n-\nu)} * (F-G)^{*\nu}(A). \tag{6}$$

While analyzing expansion's (1) asymptotic characteristics, H. Bergstrom used these conditions:

a) $V[nG^{*(n-1)} * (F - G)] < g(n), \quad \forall n > 0;$ (7)

b) $\lim_{m\to\infty} V[F^{*m} * (F - G)] = 0,$ (8)

where $V[F^{*m} * (F - G)]$ – total variation.

BERGSTROM'S THEOREM 1. *Suppose, that condition* (2) *is valid, when exists constant* $c_1 > 0$ *such, that*:

$$M_A[\Delta_n^{(\nu)}] < c_1 g^{\nu}(n) M_A[G],$$

when $\nu = 1, 2, \ldots, s$ *and* $A \in \mathrm{B}(\mathbb{R}^k)$.

BERGSTROM'S THEOREM 2. *Suppose, that conditions* (7) *and* (8) *are valid, and functions* $g(n)$ *and* $g_1(n, 1)$ *are odds*:

$$\sum_{n=1}^{\infty} \frac{1}{n} g^{s+1}(n) < \infty \quad \textit{and } M_A\big[\Delta_n^{(s+1)}\big] < \mu(A) g_1(n, s),$$

where $M_A(F) < \mu(A)$ *or* $M_A(G) < \mu(A)$, $c_2 > 0$, *then residual in expansion* (1) *meets condition*:

$$M_A\big[r_n^{(s+1)}\big] < c_2 \mu(A) g_1(n, s). \tag{9}$$

Here and further constants $c_2, c_3, \ldots$ *depends on* s, *but do not depends on* n.

BERGSTROM'S THEOREM 3. *If function* $g(n)$ *meets condition*

$$\sum_{n=1}^{\infty} \frac{1}{n} g^{s+1}(n, s) < \infty,$$

then conditions (7) *and* (8) *are necessary and sufficient, in order to be valid*:

$$V\big[\Delta_n^{(\nu)}\big] = \mathrm{O}\big(g^{\nu}(n)\big),$$

where $\nu = 2, 3, \ldots, s + 1$, s *– fixed integer and* $V[r_n^{(s+1)}] = \mathrm{O}(g^{s+1}(n))$.

Note. Condition (8) can be changed:

$$\lim_{m\to\infty} V\Big[F^{*n}(\vec{x}) * G\big((an)^{\alpha}\vec{x}\big) * \big(F(\vec{x}) - G(\vec{x})\big)\Big] = 0, \tag{10}$$

where a – constant, $a > 1$ and $\alpha = \frac{(s+1)(\lambda-2)}{2} - \frac{1}{2}$.

We have specified H. Bergstrom's 4th and 5th theorems conditions using characteristic function methods. Lets mark $\vec{\xi}$ – k-dimensional random vector, which distribution is $F(\vec{x}) = P\{\vec{\xi} < \vec{x}\}$; V – non-defective $\vec{\xi}$ second raw moment matrix; $\vec{t}V\vec{t}'$ – quadratic form; $M\vec{\xi}$ – means vector; $\vec{t}'$ – transposed vector; $\vec{\xi}$ – random vector, independent of n.

THEOREM 1. *Suppose random vector's $\vec{\xi}$ second raw moments matrix V is positively determined, $\lambda \in (2,3]$ raw moments are finite, then constant C exists and depends on k and λ, such, that*

$$\sup_B \left| C_n^\nu \Phi^{*(n-\nu)} * (F-\Phi)^{*\nu}(B\sqrt{n}) \right| \leqslant \left(C \frac{M\left[((\xi - M\xi)V^{-1}(\xi - M\xi))^{\frac{\lambda}{2}} \right]}{\sqrt{n}} \right)^\nu,$$

where $\nu = 1, 2, \ldots, s$, and B – Borel set.

For residual estimation we need:

$$\lim_{n\to\infty} \sup_B \left| F^{*n}(B\sqrt{n}) * H\big(B(\lambda(n)\sqrt{n})^s\big) * \big(F(B\sqrt{n}) - \Phi(B\sqrt{n})\big) \right| = 0, \tag{11}$$

where $\lambda(n)$ – slowly growing function, $\lim_{n\to\infty}\lambda(n) = \infty$, $H(B)$ – distribution with characteristic function $h(|\vec{t}|)$, and meets $h(|\vec{t}|) = 0$, when $|\vec{t}| \geqslant 1$, and probability function $Q(|\vec{x}|) = \mathrm{O}(e^{-\sqrt{|\vec{x}|}})$, when $|\vec{x}| \to \infty$.

THEOREM 2. *Suppose Theorem* 1 *and condition* (11) *are valid*

$$\sup_B \left| F^{*n}(B\sqrt{n}) - \Phi(A) - \sum_{\nu=1}^{s} C_n^\nu \Phi\Big(B\sqrt{\frac{n}{n-\nu}}\Big) * \big(F(B\sqrt{n}) - \Phi(B\sqrt{n})\big)^{*\nu} \right| = \mathrm{o}\big(n^{-\frac{s}{2}}\big).$$

Note 1. Condition (11) is satisfied, when Cramer condition is valid

$$\overline{\lim_{|t|\to\infty}} \left| M\mathrm{e}^{i(\vec{t},\vec{\xi})} \right| < 1.$$

Note 2. Condition (11) is satisfied, if $s = 1$ and random vector's $\vec{\xi}$ characteristic function's absolute value is 1 only when $\vec{t} = \vec{0} \in \mathbb{R}^k$.

We will use $\sqrt{n}$-row normalization:

$$f = f\Big(\frac{t}{\sqrt{n}}\Big) \quad \text{and } g = g\Big(\frac{t}{\sqrt{n}}\Big).$$

H. Bergstrom equation for characteristic functions is

$$f^n - g^n = \sum_{\nu=1}^{s} C_n^\nu g^{n-\nu}(f-g)^\nu + \hat{r}_n^{(s+1)}, \tag{12}$$

where $\hat{r}_n^{(s+1)} = \sum_{\mu=s+1}^{n} C_{\mu-1}^s f^{n-\mu}(f-g)^{s+1} g^{\mu-s-1}$.

This equation can be formalized. Lets mark $T = \{\vec{t}\colon\ f(\vec{t}) \neq 0 \text{ and } g(\vec{t}) \neq 0,\ \vec{t} \in \mathbb{R}^k\}$. When $\vec{t} \in T$, then

$$f^n = g^n \exp\left\{ n\frac{f-g}{g} - \frac{1}{2}\Big(\sqrt{n}\frac{f-g}{g}\Big)^2 + \sum_{m=3}^{\infty} \frac{(-1)^{m+1}}{m}\Big(\frac{1}{\sqrt{n}}\Big)^{m-2}\Big(\sqrt{n}\frac{f-g}{g}\Big)^m \right\}. \tag{13}$$

Expansion is valid, when $|\frac{f-g}{g}| < c < 1$.

We use V.M. Kalinin generalized Appel's polynomials

$$A_{j2}\Big(\sqrt{n}\,\frac{f-g}{g}\Big) = (-1)^j \Big(\sqrt{n}\,\frac{f-g}{g}\Big)^j \times \sum_{\nu_1+2\nu_2+\cdots+j\nu_j=j} \frac{(-1)^{3(\nu_1+\cdots+\nu_j)}}{3^{\nu_1}\ldots(j+2)^{\nu_j}\nu_1!\ldots\nu_j!}\Big(\sqrt{n}\,\frac{f-g}{g}\Big)^{2(\nu_1+\cdots+\nu_j)}$$

and we get

$$\exp\Big\{\sum_{m=3}^{\infty}\frac{(-1)^{m+1}}{m}\Big(\frac{1}{\sqrt{n}}\Big)^{m-2}\Big(\sqrt{n}\,\frac{f-g}{g}\Big)^m\Big\} = 1 + \sum_{j=1}^{\infty}\Big(\frac{1}{\sqrt{n}}\Big)^j A_{j2}\Big(\sqrt{n}\,\frac{f-g}{g}\Big). \tag{14}$$

Generalized Appel's polynomials could be expressed:

$$A_{j2}(y) = (-1)^{j+3} y^{j+2} \sum_{k=0}^{j-1} (-1)^{3k} g_{jk}^{(2)} y^{2k},$$

where

$$g_{jk}^{(2)} = \sum_{\substack{\nu_1+2\nu_2+\cdots+j\nu_j=j\\ \nu_1+\nu_2+\cdots+\nu_j=k}} \frac{1}{3^{\nu_1}\cdots(j+2)^{\nu_j}\nu_1!\ldots\nu_j!},$$

$$k = 0,1,\ldots,j-1;\ \nu_i = 0,1,2,\ldots.$$

LEMMA 1. *Inequality* $g_{jk}^{(2)} < \frac{1}{3}$ *is valid for all* $k = 0,1,\ldots,j-1$ *and* $j = 1,2,\ldots$.

LEMMA 2. *For all* $\vec{t} \in \{\vec{t}: \sqrt{n}|\frac{f-g}{g}| < 1, \vec{t} \in \mathbb{R}^k\} \cap T$ *expansion*

$$f^n = g^n \exp\Big\{n\frac{f-g}{g} - \frac{1}{2}\Big(\sqrt{n}\,\frac{f-g}{g}\Big)^2\Big\}\Big[1 + \sum_{j=1}^{\infty}\Big(\frac{1}{\sqrt{n}}\Big)^j A_{j2}\Big(\sqrt{n}\,\frac{f-g}{g}\Big)\Big]$$

is valid.

While analyzing ratio $\big(\frac{f-g}{g}\big)^{\nu}$ when $\nu = 1,2,\ldots$, we will use Scheffer's polynomials [2, p. 58].

We get formal expansion:

$$\Big(\frac{f-g}{g}\Big)^{\nu} = \int_{\mathbb{R}^k} \mathrm{e}^{-i(\vec{t},\vec{x})\frac{\tau}{\sqrt{n}}} g^{-\nu}\Big(\frac{\tau}{\sqrt{n}}\Big)\,\mathrm{d}\big(F(\vec{x}) - G(\vec{x})\big)^{*\nu}\Big|_{\tau=1}$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!}\Big(\frac{1}{\sqrt{n}}\Big)^m \int_{\mathbb{R}^k} Q_m\big((i\vec{t},\vec{x}),\nu\big)\,\mathrm{d}\big(F(\vec{x})-G(\vec{x})\big)^{*\nu}.$$

Here $Q_m((i\vec{t},\vec{x}),\nu)$ is m-multivariate polynomial.

LEMMA 3. *If*

$$\int_{\mathbb{R}^k} \mathrm{d}\big(F(\vec{x})-G(\vec{x})\big)=0,$$

$$\int_{\mathbb{R}^k} (\vec{t},\vec{x})(\,\mathrm{d}F\big(\vec{x})-G(\vec{x})\big)=0$$

and

$$\int_{\mathbb{R}^k} (\vec{t},\vec{x})^2\,\mathrm{d}\big(F(\vec{x})-G(\vec{x})\big)=0.$$

then

$$\Big(\sqrt{n}\frac{f-g}{g}\Big)^{\nu} = \sum_{m=3\nu}^{\infty} \frac{1}{m!}\Big(\frac{1}{\sqrt{n}}\Big)^{m-1} \int_{\mathbb{R}^k} Q_m\big(i(\vec{t},\vec{x}),\nu\big)\,\mathrm{d}\big(F(\vec{x})-G(\vec{x})\big)^{*\nu}.$$

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REZIUMĖ

A. Bikelis, A. Galinskis. Bergstremo asimptotiniai skleidiniai

Darbe nagrinėjamos H. Bergstremo tapatybės, panaudojant Levy–Scheffer daugianarius.

Raktiniai žodžiai: Bergstremo tapatybė, Levy–Scheffer daugianariai.