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Quasi-lattice distributions analysis

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Abstract. In this work we define quasi lattice distributions functions.

Keywords: almost periodical function, rational basis.

THEOREM 1 (Besicovitch, 1932). *In order that any trigonometrical series*

$$\sum_n a_n \mathrm{e}^{i\lambda_n x}$$

should be the Fourier series for almost periodical function it is necessary and sufficient the convergence of the series

$$\sum_n |a_n|^2.$$

From this theorem we can conclude that characteristic function $f(t)$ of the discret random variable ξ

$$f(t) = \sum_{\nu=1}^{\infty} \mathrm{e}^{it\Lambda_\nu} P\{\xi = \Lambda_\nu\}$$

is almost periodical [1–4].

The set of the values of the random variable ξ we will use the notation $\Lambda = \{\Lambda_1, \Lambda_2, \ldots\}$.

DEFINITION 1. Finite or countable set of real numbers $\beta = (\beta_1, \beta_2, \ldots, \beta_n, \ldots)$ is called linearly independent over the set of rational numbers if for every k it is true the equality

$$r_1\beta_1 + r_2\beta_2 + \cdots + r_k\beta_k = 0,$$

with $r_1, r_2, \ldots, r_n$ – rational numbers, implies that all $r_1, r_2, \ldots, r_n$ are zeroes.

Remark 1. There are no zeroes between

$$\beta_1, \beta_2, \ldots, \beta_n, \ldots$$

Remark 2. The basis $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ will be called finite, if the set has finite number of elements, otherwise β will be called infinite.

DEFINITION 2. Finite or countable set of the linearly independent real numbers $\beta_1, \beta_2, \ldots, \beta_n, \ldots$ is called by rational basis of the countable set of real numbers $\Lambda_1, \Lambda_2, \ldots, \Lambda_n, \ldots$, if the every number Λ_n may be represented by linear combination of β_j with rational coefficients, i.e.,

$$\Lambda_n = r_{i_1}^{(n)} \beta_{i_1} + \cdots + r_{i_m}^{(n)} \beta_{i_m}, \tag{1}$$

where $i_1 \neq \ldots \neq i_m$, $r_j^{(n)}$ – rational numbers.

The set of numbers $\Lambda_1, \Lambda_2, \ldots, \Lambda_n, \ldots$ has many rational bases. If it is chosen one of them, for example $\beta_1, \beta_2, \ldots \beta_n, \ldots$ then the relation (1) is unique.

PRESUMPTION 1 (Bohr, 1932). *Every set of real numbers* $\Lambda_1, \Lambda_2, \ldots, \Lambda_n, \ldots$ *has a rational basis* β [2].

Remark 3. If all $r_j = 0, \pm 1, \pm 2, \ldots$, then the basis is called by integer basis.

Remark 4 (Bohr, 1932). All finite set of real numbers $\Lambda_1, \Lambda_2, \ldots, \Lambda_N, \ldots$ has a finite integer basis $\beta_1, \beta_2, \ldots, \beta_k$, i.e.,

$$\Lambda_j = \nu_1^{(j)} \beta_1 + \nu_2^{(j)} + \cdots + \nu_k^{(j)} \beta_k,$$

where $\nu_i^{(j)} = 0, \pm 1, \pm 2, \ldots$.

If the random variable ξ is defined in the finite probability space $\{\Omega, \mathcal{A}, P\}$ (see [5], p. 32–33), then the set of his values $\Lambda_1, \Lambda_2, \ldots, \Lambda_m$ and there exists an integer basis

$$\vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_k)$$

such that

$$\Lambda_j = (\vec{a}, \vec{E}) + (\vec{\nu}_j, \vec{\beta}), \quad j = 1, 2, \ldots, m,$$

where $\vec{a} \in R^k$, $\vec{E} = (1, 1, \ldots, 1) \in R^k$, $\vec{\nu}_j = (\nu_{1j}, \ldots, \nu_{kj})$, $\nu_{ij} = 0, \pm 1, \pm 2, \ldots$, $\beta_i > 0$, $(\vec{\nu}_j, \vec{\beta})$ – a scalar product.

In the k-dimensional Euclidean space we define the random vector $\vec{\eta}$ in such way:

$$P\{\vec{\eta} = \vec{a} + \vec{\nu}_j \vec{\beta}\} = P\big\{\xi = (\vec{a}, \vec{E}) + (\vec{\nu}_j, \vec{\beta})\big\},$$

where $j = 1, 2, \ldots, m$, $\vec{\nu}_j \vec{\beta} = (\nu_{1j}\beta_1, \nu_{2j}\beta_2, \ldots, \nu_{kj}\beta_k)$.

The random vector $\vec{\eta}$ is lattice and

$$P\big\{\xi = (\vec{a}, \vec{E}) + (\vec{\nu}_j, \vec{\beta})\big\} = \frac{\beta_1, \ldots, \beta_k}{(2\pi)^{k/2}} \int_{-\frac{\pi}{\beta_1}}^{\frac{\pi}{\beta_1}} \cdots \int_{-\frac{\pi}{\beta_k}}^{\frac{\pi}{\beta_k}} \mathrm{e}^{-i(\vec{t}, \vec{a}) - i(\vec{t}, \vec{\nu}_j \vec{\beta})} M \mathrm{e}^{i(\vec{t}, \vec{\eta})} \, \mathrm{d}\vec{t},$$

where the characteristic function of $\vec{\eta}$ is $M\mathrm{e}^{i(\vec{t}, \vec{\eta})}$ [4].

DEFINITION 3. A discrete finite generalized measure μ is called m-quasi-lattice if it has the integer finite basis $\vec{\beta}$ consisting of m elements.

It is worth to arrange the elements of basis $\beta = (\ldots, \beta_m^-, \ldots, \beta_2^-, \beta_1^-, \beta_1^+, \beta_2^+, \ldots, \beta_n^+, \ldots)$ in increasing order, i.e., $\ldots < \beta_m^- < \ldots < \beta_2^- < \beta_1^- < 0 < \beta_1^+ < \beta_2^+ < \ldots \beta_n^+ \ldots$. Let $\beta^m = \beta \times \ldots \times \beta$ be a Cartesian product of m sets β. We split the set $\Lambda = \{\Lambda_1, \Lambda_2 \ldots\}$ into $\mathcal{M}(\mathcal{M} \leqslant \infty)$ disjoint sets, in such way that all numbers $r_{i_1}^{(n)}, r_{i_2}^{(n)}, \ldots, r_{i_m}^{(n)}$ in the decomposition (1) are non-zeroes (except, maybe, the case $m = 1$) for all Λ_n included in m-th subset. Let $s = s(m)$ among them are negative and $(m - s)$ positive, i.e., rewrite the equality (1) as follows:

$$\Lambda_n = \left(\vec{\beta}_m^{(s)}, \vec{r}(m)\right), \tag{2}$$

where $\vec{\beta}_m^{(s)} = (\beta_{i_1}^-, \ldots, \beta_{i_s}^-, \beta_{i_{s+1}}^+, \ldots, \beta_{i_m}^+)$, $\vec{r}(m) = (r_{i_1}^{(n)}, \ldots, r_{i_m}^{(n)}) \in \mathcal{Q}^m$, $0 \leqslant s \leqslant m$ and $m = 1, 2, \ldots, \mathcal{M}$.

Observe, that $\vec{\beta}_m^{(s)} \in \beta^m$, not all the coordinates of the vector $\vec{\beta}_m^{(s)}$ are different.

Let the set W_1 contains the numbers of form (1), where $m = 1$, i.e.,

$$W_1 = \left\{\beta_i r_i\colon \ \beta_i \in \beta, \ r_i \in \mathcal{Q}, \ i = 1, 2, \ldots\right\}.$$

The set W_2 has a form:

$$W_2 = \Big\{\beta_i r_i + \beta_k r_k\colon \ (\beta_i, \beta_k) \in \beta^2, \ \beta_i \neq \beta_k; \ i, k = 1, 2, \ldots; \\ (r_i, r_k) \in \mathcal{Q}^2; \ r_i \neq 0, \ r_k \neq 0; \ i, k = 1, 2, \ldots\Big\}.$$

Further, W_3 has a form:

$$W_3 = \Big\{\beta_{i_1} r_{i_1} + \beta_{i_2} r_{i_2} + \beta_{i_3} r_{i_3}\colon \ (\beta_{i_1}, \beta_{i_2}, \beta_{i_3}) \in \beta^3, \ \beta_{i_1} \neq \beta_{i_2} \neq \beta_{i_3}; \\ (r_{i_1}, r_{i_2}, r_{i_3}) \in \mathcal{Q}^3; \ r_{i_1} \neq 0, \ r_{i_2} \neq 0; \ r_{i_3} \neq 0\Big\}.$$

Construct the sequence of the sets $W_1, W_2, \ldots$ such that

$$\Lambda = \sum_{m=1}^{\mathcal{M}} W_m$$

and $W_i \cap W_j = \emptyset$ for $i \neq j$. Here $\mathcal{M} \leqslant \infty$.

It follows from the construction that

$$W_m = \sum_{\substack{\vec{r}(m) \in \mathcal{Q}^m \\ \vec{\beta}_m^{(s)} \in \beta^m}}^{*} \left\{\left(\vec{\beta}_m^{(s)}, \vec{r}(m)\right)\right\}.$$

Here for $m = 2, 3, \ldots$ the symbol $\sum^*$ denotes a sum over to $\vec{r}(m)$ with non-zero coordinates, and over $\vec{\beta}_m^{(s)}$ with different coordinates. The set

$$\left\{\left(\vec{\beta}_m^{(s)}, \vec{r}(m)\right)\right\}$$

consists of the only element, i.e., the scalar product of the vectors $\vec{\beta}_m^{(s)}$ and $\vec{r}(m)$:

$$\big(\vec{\beta}_m^{(s)}, \vec{r}(m)\big) = \beta_{i_1}^- r_{i_1} + \cdots + \beta_{i_s}^- r_{i_s} + \beta_{i_{s+1}}^+ r_{i_{s+1}} + \cdots + \beta_{i_m}^+ r_{i_m}.$$

Moreover, if $\vec{r}(m_1) \neq \vec{r}(m_2)$, then $(\vec{\beta}_{m_1}^{(s)}, \vec{r}(m_1)) \neq (\vec{\beta}_{m_2}^{(s)}, \vec{r}(m_2))$

We summarize the facts above in the following theorem.

THEOREM 2. *The support Λ of the generalized finite discrete measure μ_d contains the basis β such that*

$$\Lambda \subseteq \sum_{m=1}^{\mathcal{M}} \sum_{\substack{\vec{r}(m)\in Q^m \\ \vec{\beta}_m^{(s)}\in\beta^m}}^{*} \Big\{\big(\vec{\beta}_m^{(s)}, \vec{r}(m)\big)\Big\} = W(\beta). \tag{3}$$

Here $M \leqslant \infty$.

Further, β will be called basis of the measure μ_d. Assume,

$$\mathbf{E}(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1, & \text{otherwise } x \geqslant 0. \end{cases}$$

COROLLARY 1. *Let the generalized finite discrete measure μ_d has the basis β, then*

$$\mu_d\big((-\infty, x]\big) = \sum_{(\vec{\beta}_m^{(s)}, \vec{r}(m))\in W(\beta)} \mu_d\Big(\big\{\big(\vec{\beta}_m^{(s)}, \vec{r}(m)\big)\ \big\}\Big)\boldsymbol{E}\Big(x - \big(\vec{\beta}_m^{(s)}, \vec{r}(m)\big)\Big).$$

Frequently in our investigations the elements from subclasses of finite generalized measures $\mathcal{M}$ will be considered. Some of them we will define now.

DEFINITION 4. A discrete finite generalized measure μ_d is called $\mathcal{M}$-quasi-lattice if it has the integer finite basis β consisting of M elements.

Let the basis β is finite, i.e., $\beta = (\beta_1, \beta_2, \dots, \beta_{\mathcal{M}})$. In the scalar product

$$\big(\vec{\beta}, \vec{r}\big) = \beta_1 r_1 + \cdots + \beta_M r_M$$

in the representation (3) the coefficients $r_1, r_2, \dots, r_M$ are, generally speaking, the rational numbers. If the basis $\beta_1, \beta_2, \dots, \beta_M$ is integer, then $r_1, r_2, \dots, r_M = 0, \pm1, \pm2, \dots$.

In more general case, the scalar products in the representation (3) have a form

$$(\vec{\beta}, \vec{r}) = \beta_1\nu_1 + \cdots + \beta_m\nu_m + \beta_{m+1}r_{m+1} + \cdots + \beta_M r_M,$$

where $\nu_1, \dots, \nu_m = 0, \pm1, \pm2, \dots$. I.e., the basis $\beta = (\beta_1, \beta_2, \dots, \beta_{\mathcal{M}})$ consists of two sub-basises – integer basis $\beta_1, \beta_2, \dots, \beta_m$ and non-integer $\beta_{m+1}, \dots, \beta_{\mathcal{M}}$.

DEFINITION 5. Discrete finite generalized measure μ_d is called M-discrete m-quasi-lattice if its basis $\beta = (\beta_1, \ldots, \beta_m, \beta_{m+1}, \ldots, \beta_{\mathcal{M}})$ contains integer sub-basis $\beta_1, \beta_2, \ldots, \beta_m$, where m maximal number satisfying this property. Remind that $\mathcal{M} < \infty$.

Remark 5. $\mathcal{M}$-discrete $\mathcal{M}$-quasi-lattice measure μ_d has the support $W(\beta)$ of (3) form, where $\mathcal{M} < \infty$ and the coordinates of vector $\vec{r}(m)$ are integer numbers $0, \pm 1, \pm 2, \ldots$.

Note, that there exist measures with the integer infinite basis, i.e., $M = \infty$.

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REZIUMĖ

A. Bikelis. Kvazigardelinių skirstinių analizė

Darbe yra nagrinėjami tikimybiniai skirstiniai, kurių charakteringosios funkcijos yra beveik periodinės funkcijos.

Raktiniai žodžiai: beveik periodinės funkcijos, racionalioji bazė.