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On the weighted boundary value problem for systems of degenerate ordinary differential equations

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Abstract. A system of ordinary second order linear equations with a singular point is considered. The aim of this work is such that the system of eigenvectors of the matrix that couples the system of equations is not complete. That implies a matter of the statement of a weighted boundary value problem for this system. The well-posed boundary value problem is proposed in the article. The existence and uniqueness of the solution is proved.

Keywords: systems of the ordinary differential equations, irregular singularity, weighted boundary value problems.

1 Introduction

We consider a system of ordinary differential equations

$$y'' - x^{-2\alpha} A y = 0, \quad \alpha > 1, \tag{1}$$

in the interval $(0,1)$. Here $y = (y_1, y_2, \ldots, y_n)$ is an unknown column vector, and A is a constant positively defined $n \times n$ matrix. We assume that the eigenvectors of matrix A do not comprise a complete system, i.e., A is non-normal matrix. Let $h_{11}, h_{12}, \ldots, h_{1p}$ $(p < n)$ be linearly independent eigenvectors, and let each eigenvector h_{1i} $(i = \overline{1,p})$ corresponds eigenvalue λ_i and a sequence of adjoined vectors $h_{2i}, h_{3i}, \ldots, h_{s_i i}$. Since $\alpha > 1$, $x = 0$ is an irregular singular point of system of equations (1) [2, 4].

In the case where matrix A is the normal one, i.e., $p = n$, a weighted boundary value problem is considered in [3], where a boundary condition at a singular point is formulated as follows: $\lim_{x\to+0} \Phi(x) y(x) = \gamma_0$. Here Φ is a certain weighted matrix. If $p < n$, then the problem with such weighted condition is incorrect. In the present work we analyze a boundary value problem of the following shape:

$$\lim_{x\to+0} \Psi(x)\big[S^{-1} y(x) - \Omega(x)\gamma_0\big] = \gamma_0, \qquad y(1) = \gamma_1. \tag{2}$$

Here S is a matrix the columns of which are comprised of all the linearly independent eigen- and adjoined vectors of matrix A, $\gamma_i = (\gamma_{i1}, \gamma_{i2}, \ldots, \gamma_{in})$, $i = 0, 1$, are arbitrary given column vectors,

$$\Psi(x) = \operatorname{diag}\{\psi_1(x)E_{s_1}, \psi_2(x)E_{s_2}, \dots, \psi_p(x)E_{s_p}\},$$
$$\psi_i(x) = x^{-\frac{\alpha}{2}} \exp\left\{-\frac{\sqrt{\lambda_i}}{\alpha-1}x^{1-\alpha}\right\},$$

where E_{s_i} is an s_i-order unit matrix, $\Omega(x)$ is a certain block matrix to be defined late on.

Our aim is to select the matrix $\Omega(x)$ in the first of conditions (2) so that problem (1), (2) would have a unique solution, and to prove the existence of a solution of this problem.

2 Statement and solution of problem (1), (2)

We look for a solution y of problem (1), (2) in the form

$$y(x) = Su(x). \tag{3}$$

Then, with respect to an unknown vector u we get the following differential problem:

$$u'' - x^{-2\alpha}Ju = 0, \quad 0 < x < 1, \tag{4}$$
$$\lim_{x\to+0} \Psi(x)\big[u(x) - \Omega(x)\gamma_0\big] = \gamma_0, \qquad u(1) = \tilde{\gamma}_1. \tag{5}$$

Here $\tilde{\gamma}_1 = S^{-1}\gamma_1$, $J = S^{-1}AS = (J_{\lambda_1}, J_{\lambda_2}, \dots, J_{\lambda_p})$ is a canonical Jordan form of matrix A, where J_{λ_i} is s_i-order lower Jordan block with the parameter λ_i in the diagonal. We assume that all the eigenvalues are real and matrix $\Omega(x)$ is of a block shape $\Omega(x) = \operatorname{diag}\{\Omega_{s_1}(x), \Omega_{s_2}(x), \dots, \Omega_{s_p}(x)\}$, where $\Omega_{s_i}(x)$ is a some $s_i\times s_i$ matrix. Then problem (4), (5) splits into p separate problems:

$$\frac{d^2u^{(i)}}{dx^2} - x^{-2\alpha}J_{\lambda_i}u^{(i)} = 0, \tag{6}$$
$$\lim_{x\to+0} \psi_i(x)\big[u^{(i)}(x) - \Omega_{s_i}(x)\gamma_0^{(i)}\big] = \gamma_0^{(i)}, \qquad u^{(i)}(1) = \tilde{\gamma}_0^{(i)}, \tag{7}$$

$i = \overline{1,p}$, where $u^{(i)} = (u_{\sigma_i+1}, \dots, u_{\sigma_i+s_i})$, $\gamma_0^{(i)} = (\gamma_{0,\sigma_i+1}, \dots, \gamma_{0,\sigma_i+s_i})$, $\tilde{\gamma}_0^{(i)} = (\tilde{\gamma}_{1,\sigma_i+1}, \dots, \tilde{\gamma}_{1,\sigma_i+s_i})$, and $\sigma_i = s_1 + s_2 + \dots + s_{i-1}$.

In order to avoid complicated notation, we introduce a differential operator $l := \frac{d^2}{dx^2} - \lambda x^{-2\alpha}$, $\lambda = \text{const} > 0$, and instead of the system (4), let us consider a system

$$l(u_1) = 0, \quad l(u_2) = x^{-2\alpha}u_1, \quad \dots, \quad l(u_s) = x^{-2\alpha}u_{s-1}, \tag{8}$$

that corresponds to the s-order lower Jordan block J_λ with the parameter λ in the diagonal.

Note that the functions

$$u_{01}(x) = c_1\sqrt{x}K_{\frac{1}{2(\alpha-1)}}\left(\frac{\sqrt{\lambda}}{\alpha-1}x^{1-\alpha}\right), \qquad u_{02}(x) = c_2\sqrt{x}I_{\frac{1}{2(\alpha-1)}}\left(\frac{\sqrt{\lambda}}{\alpha-1}x^{1-\alpha}\right)$$

are linearly independent solutions of the first equation $l(u_1) = 0$. (Here c_1 and c_2 are arbitrary constantse, and $K_v(x)$ and $I_v(x)$ are modified Bessel functions.) According to asymptotic exspresions of functions $K_v(x)$ and $I_v(x)$, as $x \to +\infty$ [1], by taking $c_1 = \sqrt{2(\alpha-1)/(\pi\sqrt{\lambda})}$, $c_2 = \sqrt{(\alpha-1)/(2\pi\sqrt{\lambda})}$, we obtain that

$$u_{01}(x) \sim x^{\frac{\alpha}{2}} \exp\bigl\{-\xi(x)\bigr\}, \qquad u_{02}(x) \sim x^{\frac{\alpha}{2}} \exp\bigl\{\xi(x)\bigr\}, \tag{9}$$

$$u'_{01}(x) \sim \sqrt{\lambda} x^{-\frac{\alpha}{2}} \exp\bigl\{-\xi(x)\bigr\}, \qquad u'_{02}(x) \sim -\sqrt{\lambda} x^{-\frac{\alpha}{2}} \exp\bigl\{\xi(x)\bigr\}, \tag{10}$$

as $x \to +0$, where $\xi(x) = \sqrt{\lambda} x^{1-\alpha}/(\alpha-1)$. So $u_{01}(x) \to 0$, and $u_{02}(x) \to +\infty$, as $x \to +0$. Moreover, the properties of Bessel function $K_v(x)$ yield $u_{01}(1) \neq 0$.

We will use below more convenient solution $\tilde{u}_{02}(x) = u_{02}(x) - u_{01}^{-1}(1)u_{02}(1)u_{01}(x)$ of equation $l(u_1) = 0$, which satisfies the condition $\tilde{u}_{02}(1) = 0$, obviously, and has asymptotic expressions, as $x \to +0$, quite coincident with the asymptotic expressions (9) and (10) of the solution $u_{02}(x)$. For convenience, everywhere abowe we shall write $u_{02}(x)$ instead of $\tilde{u}_{02}(x)$.

Denote

$$\varGamma(x,t) = W^{-1}\bigl[u_{02}(x)u_{01}(t) - u_{01}(x)u_{02}(t)\bigr],$$

$$G(x,t) = W^{-1}\begin{cases} u_{02}(x)u_{01}(t), & 0 < t \leqslant x, \\ u_{01}(x)u_{02}(t), & x \leqslant t \leqslant 1, \end{cases}$$

where $W = u_{01}(x)u'_{02}(x) - u'_{01}(x)u_{02}(x)$ is a Wronskian. We define onto the set of functions $\varphi \in C(0,1]$ the integral operators

$$\mathbf{K}(\varphi)(x) = \int_x^1 \varGamma(x,t)t^{-2\alpha}\varphi(t)\,dt, \qquad \mathbf{L}(\varphi)(x) = \int_0^1 G(x,t)t^{-2\alpha}\varphi(t)\,dt,$$

as well as their integer positive powers

$$\mathbf{K}^k(\varphi)(x) = \mathbf{K}\bigl(\mathbf{K}^{k-1}(\varphi)\bigr)(x), \qquad \mathbf{K}^0(\varphi)(x) \equiv \varphi(x),$$

$$\mathbf{L}^k(\varphi)(x) = \mathbf{L}\bigl(\mathbf{L}^{k-1}(\varphi)\bigr)(x), \qquad \mathbf{L}^0(\varphi)(x) \equiv \varphi(x).$$

Let us compose the functions $k_{ij}(x)$ and $l_{ij}(x)$, which are identically equal to zero for $1 \leqslant i < j \leqslant s$ and are defined by the formulae $k_{ij}(x) = \mathbf{K}^{i-j}(u_{02})(x)$, $l_{ij}(x) = \mathbf{L}^{i-j}(u_{01})(x)$ for $1 \leqslant j \leqslant i \leqslant s$, and let us compose triangular $s \times s$ matrices $\mathcal{K}_s(x) = (k_{ij}(x))$, $\mathcal{L}_s(x) = (l_{ij}(x))$. Since $l(\mathbf{K}^k(\varphi)) = x^{-2\alpha}\mathbf{K}^{k-1}(\varphi)(x)$, $l(\mathbf{L}^k(\varphi)) = x^{-2\alpha}\mathbf{L}^{k-1}(\varphi)(x)$, $k = \overline{1,s}$, each of these matrices is the matrix solution of system (8). It is easy to verify that $\mathcal{K}_s(1) = \varTheta$, $\mathcal{L}_s(1) = E_s$. (Here $\varTheta$ is a zero matrix.) Let $C_s^{(i)}$, $i = 1,2$, be arbitrary s-dimensional column vectors. Then

$$\mathcal{U}(x) = \mathcal{K}_s(x)C_s^{(1)} + \mathcal{L}_s(x)C_s^{(2)} \tag{11}$$

is the general solution of the system of equations (8), evidently.

Lemma 1. *Let $\eta_k(x) = x^{(1-\alpha)k}/(k!(2\sqrt{\lambda})^k(\alpha-1)^k)$. If $\lambda > 0$, then the asymptotic relations*

$$\mathbf{K}^k(u_{02})(x) \sim \eta_k(x)u_{02}(x), \qquad \mathbf{L}^k(u_{01})(x) \sim (-1)^k\eta_k(x)u_{01}(x), \tag{12}$$

as $x \to +0$, hold $\forall k = \overline{1,s}$.

Proof. Firstly, we prove the first of relations (12). It follows from formulas (9), (10), that $W = -2\sqrt{\lambda} + o(1)$, $x \to +0$. Since Wronskian W must be constant, we get that $W = -2\sqrt{\lambda}$. Then, as $k = 1$, we obtain due to asymptotic relations (9), (10) by the l'Hôpital rule that

$$\lim_{x\to+0} \frac{\mathbf{K}(u_{02})(x)}{x^{1-\alpha}u_{02}(x)} = \frac{1}{2\sqrt{\lambda}} \lim_{x\to+0} \frac{(\int_x^1 \varGamma(x,t)u_{02}(t)t^{-2\alpha}\,dt)'}{(x^{1-\alpha}u_{02}(x))'} = \frac{1}{2\sqrt{\lambda}(\alpha-1)}.$$

Let the first of relation (12) holds, as $k = \sigma - 1$, i.e.,

$$\mathbf{K}^{\sigma-1}(u_{02})(x) \sim \eta_{\sigma-1}(x)u_{02}(x), \quad x \to +0. \tag{13}$$

According to the definition of the powers of operator $\mathbf{K}$ we obtain that

$$\lim_{x\to+0} \frac{\mathbf{K}^{\sigma}(u_{02})(x)}{x^{1-\alpha}\mathbf{K}^{\sigma-1}(u_{02})(x)} = \lim_{x\to+0} \frac{\int_x^1 \varGamma(x,t)\mathbf{K}^{\sigma-1}(u_{02})(t)t^{-2\alpha}\,dt}{2\sqrt{\lambda}x^{1-\alpha}\mathbf{K}^{\sigma-1}(u_{02})(x)}.$$

After inserting the right-hand side of relation (13), that is true according to the premise of induction, into the denominator of the latter fraction, in accordance with the l'Hôpital rule we obtain that

$$\lim_{x\to+0} \frac{\mathbf{K}^{\sigma}(u_{02})(x)}{x^{1-\alpha}\mathbf{K}^{\sigma-1}(u_{02})(x)} = \frac{1}{2\sqrt{\lambda}(\alpha-1)},$$

i.e., the first part of formula (12) is valid for $k = \sigma$. Consequently, the first one of the asymptotic relations (12) holds $\forall k = \overline{1,s}$.

The second of the asymptotic relation (12) is proved analogously.

Let us define a function $\psi(x) = x^{-\frac{\alpha}{2}}\exp\{-\xi(x)\}$. It is easily seen that $\lim_{x\to+0}\psi(x) = 0$.

Corollary 1. *It follows from asymptotic relations* (9), (10), *and* (12) *that*

$$\lim_{x\to+0} \mathbf{L}^k(u_{01})(x) = 0 \quad \forall k = \overline{1,s} \quad \textit{and} \quad \lim_{x\to+0} \psi(x)\mathbf{K}^k(u_{02})(x) = \begin{cases} 1, & k = 1, \\ +\infty, & k = \overline{2,s}. \end{cases}$$

Let $\varOmega_s(x) = \mathcal{K}_s(x) - u_{02}(x)E_s$ and let us consider the following boundary value problem to the system (8):

$$\lim_{x\to+0} \psi(x)E_s\big[u(x) - \varOmega_s(x)\alpha\big] = \alpha, \qquad u(1) = \beta, \tag{14}$$

here α, β are arbitrary s-dimensional constant vectors.

Taking in (11) $C_s^{(1)} = \alpha$, we get according to Lemma 1 and Corollary 2 that

$$\lim_{x\to+0} \psi(x)E_s\big[u(x) - \varOmega_s(x)\alpha\big] = \lim_{x\to+0} \psi(x)\big[u_{02}(x)E_s\alpha + L_s(x)C_s^{(2)}\big] = \alpha,$$

i.e., the solution $\mathcal{K}_s(x)\alpha + \mathcal{L}_s(x)C_s^{(2)}$ of system (8) satisfies the first of conditions (14). If $C_s^{(2)} = \beta$, then the solution

$$u(x) = \mathcal{K}_s(x)\alpha + \mathcal{L}_s(x)\beta \tag{15}$$

satisfies the second condition of (14), evidently. Thus, the vector function (15) in the solution to the boundary value problem (8), (14).

Let $u_{01}^{(i)}$, $u_{02}^{(i)}$ be solutions of the equation $u'' - \lambda_i x^{-2\alpha} u = 0$ for which asymptotic expressions (9), (10) hold, as $\lambda = \lambda_i$, and let $u_{02}^{(i)}(1) = 0$ and $\mathcal{K}_{s_i}(x)$, $\mathcal{L}_{s_i}(x)$, $\Omega_{s_i}(x)$ be matrices formed just like the matrices $\mathcal{K}_s(x)$, $\mathcal{L}_s(x)$, $\Omega_s(x)$, taking λ_i instead of λ. Then, as is shown before, $u^{(i)}(x) = K_{s_i}(x)\gamma_0^{(i)} + L_{s_i}(x)\tilde{\gamma}_1^{(i)}$ is the solution of shape (15) to the boundary value problem (6), (7). Obviously, vector column

$$u(x) = \big(u^{(1)}(x), u^{(2)}(x), \dots, u^{(p)}(x)\big) \tag{16}$$

is the solution to the boundary value problem (4), (5). Thus, the matrix $\Omega(x)$ in conditions (2) and (5) should be defined as follows:

$$\Omega(x) = \operatorname{diag}\big\{\Omega_{s_1}(x), \Omega_{s_2}(x), \dots, \Omega_{s_p}(x)\big\}.$$

Then the solution of the boundary problem (1), (2) can be expressed by equality (3), the vector function $u(x)$ of which is defined by formula (16).

We shall show that the solution to the boundary value problem (1), (2) is unique. To this end, we shall prove an auxiliary statement.

Lemma 2. *Let a matrix A be positively defined and let $y(x) = (y_1(x), y_2(x), \dots, y_n(x))$ be the solution of system* (1). *Then the function $\rho(x) = \|y(x)\|^2 = \sum_{i=1}^n y_i^2(x)$ cannot attain a positive maximum on the interval* $(0,1)$.

Proof. Note that $\rho' = 2(y', y)$, $\rho'' = 2(y'', y) + 2\|y'\|^2$. Then, multiplying the system (1) by y in the scalar way, we get that $\rho'' - 2\|y'\|^2 - 2x^{-2\alpha}(Ay, y) = 0$. Hence, it follows that $\rho''(x) > 0$, $x \in (0,1)$. Hence,if $\rho(x)$ attains a positive maximum at some point $x_0 \in (0,1)$, then $\rho''(x) \geqslant 0$, which is impossible. So $\rho(x)$ cannot acquire the positive maximum on the interval $(0,1)$.

Let $\tilde{y}_1, \tilde{y}_2$ be the solutions of problem (1), (2). Then the vector function $y = \tilde{y}_1 - \tilde{y}_2$ satisfies the system of differential equations (1) and the homogeneous boundary conditions

$$\lim_{x\to+0} \Psi(x) S^{-1} y(x) = 0, \qquad y(1) = 0.$$

Then it follows from the first of these conditions as well as from Lemma 1 that $\lim_{x\to+0} \|y(x)\| = 0$. Since $\|y(1)\| = 0$, according to Lemma 2, $\|y(x)\| = 0$ in the interval $(0,1)$. Consequently, $\tilde{y}_1(x) \equiv \tilde{y}_2(x)$.

Thus, there holds

Theorem 1. *Let matrix A be not a normal one and its eigenvalues be positive. Then, the boundary value problem* (1), (2) *is well-posed, i.e., there exists its unique solution. The solution can be expressed by equality* (3), *in which $u(x)$ is defined by formula* (16).

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REZIUMĖ

Apie išsigimstančių paprastųjų diferencialinių lygčių sistemų svorinius kraštinius uždavinius

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Nagrinėjama antrosios eilės tiesinių paprastųjų diferencialinių lygčių sistema su ypatinguoju tašku. Darbo esmė yra ta, kad matricos, siejančios lygčių sistemą, tikrinių vektorių sistema nėra pilna. Tai sudaro keblumų korektiškai suformuluoti šios lygčių sistemos svorinio tipo kraštinį uždavinį. Straipsnyje pasiūlyta korektiška svorinio kraštinio uždavinio formuluotė ir įrodytas suformuluotojo uždavinio sprendinio egzistavimas bei vienatis.

Raktiniai žodžiai: paprastųjų diferencialinių lygčių sistemos, ireguliarioji ypatuma, svoriniai kraštiniai uždaviniai.