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Uniform distribution in the n-dimensional torus

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Abstract. Limit distribution of endomorphisms of the n-dimensional torus is examined. The obtained result generalizes earlier results of the authors.

Keywords: uniform distribution, endomorphisms of n-dimensional torus.

1 Notations and results

Let $\Omega = \Omega_n$ be the n-dimensional torus

$$\mathbf{x} = (x_1, \dots, x_n), \quad 0 \leqslant x_i < 1,$$

with the coordinate-wise summation modulo 1. We define the rectangle $\Pi = [a, b] \times [c, d]$ and the functions $\varphi_i(x, y)$, $i = 1, \dots, n-2$, presenting the surfaces in the Euclidean three-dimensional space $\sigma_i = (x, y, \varphi_i(x, y))$, $i = 1, \dots, n-2$, with mixed partial derivatives on Π.

If Π', $\Pi' \subset \Pi$, the measure $\mu_{\sigma_i}(\Pi')$ is expressed by

$$\mu_{\sigma_i}(\Pi') = \iint\limits_{\Pi'} \sqrt{1 + \left(\varphi'_{ix}\right)^2 + \left(\varphi'_{iy}\right)^2}\, dx\, dy,$$

and μ_{σ_i} is absolutely continuous with respect to the Lebesgue measure on Π.

We examine the endomorphisms of the torus Ω defined by the non-singular matrices V with integer elements by

$$T\mathbf{x} = \mathbf{x}V \pmod 1.$$

In this case the vector $(x, y, \varphi_1(x, y), \dots, \varphi_{n-2}(x, y))$ defines the surface Γ in $\mathbf{R}^n$. Let

$$K_i = \frac{\varphi''_{ix^2} \cdot \varphi''_{iy^2} - (\varphi''_{ixy})^2}{(1 + (\varphi'_{ix})^2 + (\varphi'_{iy})^2)^2}, \quad i = 1, \dots, n-2,$$

K_i being the Gaussian (total) curvature of components of the surface.

We suppose that the partial derivatives of the third order of functions $\varphi_i(x, y)$, $(x, y) \in \Pi$, exist.

Theorem 1. *Let the surfaces σ_i have the positive curvatures K_i for all $(x,y) \in \Pi$. Let the characteristic polynomial of the matrix V with different real irrational roots is irreducible over the field of rational numbers. Then for almost all points $(x,y) \in \Pi$ the sequence*

$$\big(x, y, \varphi_1(x,y), \ldots, \varphi_{n-2}(x,y)\big) \cdot V^k, \quad k = 1, 2, \ldots,$$

is uniformly distributed on the unit cube $\bar{\Omega}_n = [0,1]^n$ of the space $\mathbf{R}^n$.

In 1987 D. Moskvin [5] investigated mappings of the torus Ω_2. Later these results were extended by the authors [2] to the class of special surfaces on Ω_4. The above theorem generalizes these results to the case of $n-2$ surfaces on Ω_n.

2 Auxiliary statements

The proof of the theorem is based on Lemmas 1–5 proved in [4]. The proof also makes use of the Korobov function class (see [1]) $E_n^\alpha(c)$, $\alpha > 1$, $c > 0$, concerning Fourier coefficients of functions.

Lemma 1. *Let $\varepsilon_N^1, \ldots, \varepsilon_N^n$ be real numbers such that $\delta = \delta_N = \max_i |\varepsilon_N^i| \to 0$, $N \to \infty$. Let $\varrho_1, \ldots, \varrho_n$ be algebraic numbers linearly independent over the field of rational numbers. Then for $f \in E_n^\alpha(c)$ the following quadrature formula*

$$\frac{1}{N} \sum_{k=1}^{N} f\big(\{k(\varrho_1 + \varepsilon_N^1)\}, \ldots, \{k(\varrho_n + \varepsilon_N^n)\}\big) = \int_{\bar{\Omega}_n} f(\mathbf{x})\, d\mathbf{x} + O\bigg(\frac{c}{N} + cN\delta\frac{\alpha-1}{1+\varepsilon}\bigg)$$

holds, where $\varepsilon > 0$ is an arbitrary fixed number, the constant in O depends on α, ε, n and arithmetic properties of $\varrho_1, \ldots, \varrho_n$.

We denote

$$a_i = \frac{\partial \varphi_i(x_0, y_0)}{\partial x}, \quad b_i = \frac{\partial \varphi_i(x_0, y_0)}{\partial y}, \quad i = 1, \ldots, n-2,$$

for the fixed point $(x_0, y_0) \in \Pi$ and

$$G_i(x,y) = \left(\frac{\partial \varphi_i}{\partial x} - a_i\right)^2 + \left(\frac{\partial \varphi_i}{\partial y} - b_i\right)^2, \quad i = 1, \ldots, n-2.$$

Lemma 2. *Let the total curvature K_i satisfy the condition $K_i \geqslant \xi_i > 0$, $i = 1, \ldots, n-2$. If $\max(|x - x_0|, |y - y_0|) \leqslant \delta$ then there exist constants $c_i > 0$, such that*

$$G_i(x,y) \geqslant c_i \min\big(|x - x_0|^2, |y - y_0|^2\big), \quad i = 1, \ldots, n-2,$$

for certain (x_0, y_0).

Lemma 3. *Let a_i and b_i be such that*

$$G_i(x,y) \geqslant d_i^2 > 0, \quad i = 1, \ldots, n-2,$$

for $(x,y) \in \Pi$, d_i being fixed constants. There exists a quadratic net with lines parallel to coordinate axes and with sides of fixed length such that at least one of the assertions

$$\left|\frac{\partial \varphi_i}{\partial x} - a_i\right| \geqslant \frac{d_i}{2\sqrt{2}}, \qquad \left|\frac{\partial \varphi_i}{\partial y} - b_i\right| \geqslant \frac{d_i}{2\sqrt{2}}$$

holds for $(x,y) \neq (x_0, y_0)$.

Lemma 4. *Let $\psi_1(t), \dots, \psi_m(t)$ be any m times differentiable functions. Let $\mathbf{a} = (a_1, \dots, a_m)$ denote a nonzero vector and the Wronskian $W[\psi_1, \dots, \psi_m] > 0$ in the interval $t \in [a, b]$. If the function $g(t) = a_1\psi_1(t) + \dots + a_m\psi_m(t)$ is equal to zero for $t = t_0$, then there exist λ_1 and λ_2 such that $|g(t)| > \lambda_1|t - t_0|^{m-1}$ for $|t - t_0| \leqslant \lambda_2$, $\lambda_1 > 0$, $\lambda_2 > 0$.*

Lemma 5. *Let θ be a real root of the characteristic polynomial of the matrix $V = \|a_{ik}\|$ and $\mathbf{w} = (w_1, \dots, w_n)$ be the eigenvector corresponding to θ. If the polynomial is irreducible over the field of rational numbers, then the relation*

$$m_1 w_1 + \dots + m_n w_n = 0$$

is possible if $m_i = 0$ for all i.

Proof of Theorem 1. We suppose that all eigenvalues θ_i of matrix V are positive, different and $\theta_1 > \theta_2 > \dots > \theta_n$. The corresponding eigenvectors $\mathbf{w}_i = (w_{i1}, \dots, w_{in})$ form the base in $\mathbf{R}^n$. So any arbitrary vector $\mathbf{x} = (x_1, \dots, x_n)$ multiplied by V^m can be expressed as follows

$$\mathbf{x}V^m = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} v_{ij} x_j \right) \theta_i^m \mathbf{w}_i,$$

v_{ij} being real numbers completely defined by the matrix W and independent of numbers m.

We introduce the linear form $L_i(\mathbf{x}) = \sum_{j=1}^{n} v_{ij} x_j$. It follows from Lemma 5 that the inner product $\omega = \mathbf{w}_1 \cdot \mathbf{m} = \sum w_{1j} m_j \neq 0$ for $\mathbf{m} \neq \mathbf{0}$. Therefore for the function

$$f(x, y) = \frac{(x, y, \varphi_1(x, y), \dots, \varphi_{n-2}(x, y)) \cdot V^m \cdot \mathbf{m}}{\theta^m \cdot \mathbf{w}_1 \cdot \mathbf{m}}$$

we get the following representation

$$f(x, y) = L_1\bigl(x, y, \varphi_1(x, y), \dots, \varphi_{n-2}(x, y)\bigr) + \left(\frac{\theta_1}{\theta_{1+k}} \right)^m \frac{f_1(x, y)}{\omega},$$

where $f_1(x, y)$ is bounded on Π together with its mixed derivatives of the third order.

Let us examine separately two cases:

(1°) $v_{1j} \neq 0$ for some $j \geqslant 3$;

(2°) $v_{1j} = 0$ for all $j \geqslant 3$.

Suppose that $v_{1j_0} \neq 0$ (case 1°). Than there exists a vector $\mathbf{m}$ such that

$$\frac{\theta_{j_0}}{\theta_{j_0+k}} \frac{f_1(x, y)}{\omega} \leqslant \frac{1}{\ln m}, \quad m \geqslant 2.$$

This estimate gives expression for the total curvature K_f of the surface $z = f(x, y)$ as follows:

$$K_f = \frac{(\sum v_{1k}\varphi''_{kx^2})(\sum v_{1k}\varphi''_{ky^2}) - (\sum v_{1k}\varphi''_{kxy})^2}{(1 + (\sum v_{1k}\varphi'_{kx})^2 + (\sum v_{1k}\varphi'_{ky})^2)^2} + O\biggl(\frac{1}{\ln m}\biggr),$$

where the sums are taken over k, $k = 3, \dots, n-2$.

In this case the boundedness of derivatives φ'_{j_0x} and φ'_{j_0y} implies $K_f \geqslant c_1$ for sufficiently large m. So the rest of the proof is based on Lemmas 1–5 and coincide with that in [4].

In case 2° we use notations $a = v_{11}$, $b = v_{12}$ and examine the integral

$$J_m = \iint_{\Pi'} \exp\bigl(2\pi i\bigl(\mathbf{m}\cdot\bigl(\boldsymbol{\xi}\cdot V^m\bigr)\bigr)\bigr)\,dx\,dy$$

with $\boldsymbol{\xi} = (x, y, \varphi_1(x,y), \ldots, \varphi_{n-2}(x,y))$. The change of variables $u = x$, $v = \omega\theta_1^m(ax+by)$ implies that

$$J_m = \frac{1}{\omega b\theta_1^m}\int_0^1 dx\int_{v_1}^{v_2}\exp\left(2\pi i\left(v + \theta_1^m f_1\left(u, \frac{v-u_1}{ab\theta_1^m}\right)\right)\right)dv$$

with $v_1 = \omega\theta_1^m au$, $v_2 = \omega\theta_1^m bu$, and $u_1 = \frac{au}{b\theta_1^m}$.

The function $f_2(x) = \theta_2^m f_1(u,x)$ satisfies the Lipshitz condition $|f_2(x) - f_2(x')| \leqslant \theta_2^m|x - x'|$ and the inner integral may be estimated by

$$\int_0^1 \mathrm{e}^{2\pi iv}\left(\int_a^b \exp\bigl(2\pi i f_1(u,x)\bigr)\,dx + O\left(\frac{\theta_2^m}{\omega\theta_1^m}\right)\right)dv = O\left(\frac{1}{\omega}\left(\frac{\theta_2}{\theta_1}\right)^m\right).$$

Similar to the proof of Theorem in [4] we get the limit

$$\lim_{N\to\infty}\sum_{k=1}^N \exp\bigl(2\pi i\bigl(\mathbf{m}\cdot\boldsymbol{\xi}V^k\bigr)\bigr) = 0, \quad \mathbf{m}\neq\mathbf{0},$$

which proves the theorem. □

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REZIUMĖ

Tolygus pasiskirstymas n-mačiame tore

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Tiriamas tam tikrų n-mačio toro endomorfizmų generuotų sekų ribinis pasiskirstymas. Gautas rezultatas apibendrina ankstesnius autorių rezultatus.

Raktiniai žodžiai: tolygus pasiskirstymas, n-mačio toro endomorfizmai.