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The representation formula for solutions of some class Hamilton–Jacobi equations

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Abstract. The lower semicontinious solutions of Hamilton–Jacobi equation are contructed by Hopf formula, when hamiltonian is maximum of linear functions.

Keywords: Hamilton-Jacobi equations, lower semicontinious solutions, Hopf formula.

1 Introduction

We consider the Cauchy problem for Hamilton–Jacobi equation of the form

$$u_t + H(u_x) = 0, \tag{1}$$
$$u(0,x) = \varphi(x) \tag{2}$$

in domain $S = \{(t,x)\colon\ t > 0,\ x \in R^n\}$ with the lower semicontinious (lsc) initial function φ.

For Hamilton H is convex with respect to u_x, A. Douglis, S.N. Kruzkov first defined the notion of the generalized (semiconcave) solution of (1), (2).

Definition 1. The Lipschits continuous function $u(t,x)$ in S_T is called the generalized (semiconcave) solution of (1), (2) if $u(t,x)$ solves (1) a.e. on S_T, satisfies (2), and for $\forall l \in R^n$, $\exists C_\delta \succ 0$, that the inequality

$$u(t,x+l) - 2u(t,x) + u(t,x-l) \leqslant C_\delta |l|^2, \tag{3}$$

holds, when $(t,x) \in S_T^\delta = \{(t,x)\colon\ 0 < \delta \leqslant t \leqslant T,\ x \in R^n\}$.

E. Hopf gave [2] the representation formula for the semiconcave (1), (2) solutions.

Theorem 1. *Suppose $H(p)$ is convex and satisfies the coercivity condition*

$$\lim_{|p|\to\infty} \frac{H(p)}{|p|} = +\infty. \tag{4}$$

Let $\varphi \in Lip(R^n)$, then the semiconcave solution of (1), (2) can be represented by formula

$$u(t,x) = \min_{\xi \in R^n} \left[\varphi(\xi) + t\Phi\left(\frac{x-\xi}{t}\right)\right], \tag{5}$$

where $\Phi(q) = \sup_{p \in R^n}[(p,q) - H(p)]$ is the Legrendre tranform of $H(p)$.

If is $H(p)$ strictly convex, S.N. Kruzkov proved [3], that formula (5) gives the semiconcave solution, when $\varphi(x)$ is bounded and lsc on R^n. The solution in this case satisfies initial condition in the sense

$$\lim_{t\to 0} u(t,x) = \varphi(x).$$

The function

$$F(t,x,\xi) = t\Phi\left(\frac{x-\xi}{t}\right)$$

satisfied the initial condition

$$F(0,x,\xi) = \begin{cases} 0, & x=\xi, \\ +\infty, & x\neq\xi \end{cases} \tag{6}$$

is called the fundamental solution of (1).

For example

$$u_t + a|u_x|^2 = 0, \tag{7}$$

$a \succ 0$, the Legrendre transform of $H(p) = a|p|^2$ is $\Phi(q) = \frac{|q|^2}{4a}$, and the fundamental solution of (7) is

$$F(t,x,\xi) = \frac{|x-\xi|^2}{4at}.$$

2 The calculation of fundamental solutions

In order to define a function $\Phi(q)$ we need to solve the equation

$$x = H_p\varphi'(y)t + y$$

with respect y. In general we can not do it. It can be done when hamiltonian has the form

$$H(p) = \max_{i=1,\dots,m}\big((a^i,p)+b_i\big), \tag{8}$$

where $a^i, p \in R^n$, $b_i \in R$. We define the fundamental solution and prove the representation formula (5) for solutions of

$$u_t + \max_{i=1,\dots,m}\big((a^i,u_x)+b_i\big) = 0. \tag{9}$$

Notice, that the coercivity condition (4) for the hamiltonian (8) is not satisfied.

Let $x \in R$. For the linear equation

$$u_t + a_i u_x + b_i = 0,$$

where $a_i = \mathrm{const}$, the Legrendre tranform of $H(p) = a_i p + b_i$ is

$$\Phi(q) = \begin{cases} -b_i, & q = a_i, \\ +\infty, & q\neq a_i, \end{cases}$$

and the fundamental solution

$$F(t,x,\xi) = \begin{cases} -b_i t, & \xi = x - a_i t, \\ +\infty, & \xi \neq x - a_i t. \end{cases}$$

The solution can be represented by formula

$$u(t,x) = \min_{\xi \in R^n} \bigl[\varphi(\xi) + F(t,x,\xi)\bigr] = \varphi(x - a_i t) - b_i t.$$

This solution does not satisfied semiconcave property (3), when $\varphi(x) = |x|$. Thus we need to consider the other class of generalized solutions of (1), (2), which has been defined in [1].

Definition 2. A lsc function u on S with values in $R \cup \{+\infty\}$ is a lsc solution of (1), (2), if

$$p_t + H(p_x) = 0,$$

for all $(p_t, p_x) \in D^- u(t,x)$ (superdifferential), when $u(t,x) < +\infty$, and

$$\lim_{(t,y)\to(+0,x)} u(t,y) = \varphi(x).$$

We use the theorem which was proved in this paper.

Theorem 2. *Let* $\varphi : R^n \to (-\infty, +\infty]$ *be lsc and satisfy*

$$\varphi(x) \geqslant -C\bigl(|x| + 1\bigr), \quad C > 0,\ x \in R.$$

Let H be finite, continuous and convex. Then u defined by formula (5) *is the unique lsc solution of* (1), (2), *that is bounded from below by a function of linear growth.*

For the hamiltonians (8), suppose $a_{i+1} \succ a_i$, the Legrendre transform is

$$\Phi(q) = \begin{cases} \frac{b_i - b_{i+1}}{a_{i+1} - a_i}(q - a_i) - b_i, & q \in [a_i, a_{i+1}], \\ +\infty, & q \prec a_1,\ q \succ a_m. \end{cases}$$

Then the function

$$F(t,x,\xi) = \begin{cases} \frac{b_i - b_{i+1}}{a_{i+1} - a_i}(x - \xi - a_i t) - b_i t, & \xi \in [x - a_{i+1}t, x - a_i t],\ i = 1, \dots, m-1, \\ +\infty, & \xi \prec x - a_m t,\ \xi \succ x - a_1 t, \end{cases}$$

is convex, satisfies a.e. (9) in $\{(t,x)\colon x \in [\xi + a_1 t, \xi + a_m t]\}$and the initial condition (6), thus, from the last theorem we have, that it is the unique fundamental solution of (9).

Example 1. Suppose we have the Cauchy problem

$$u_t + |u_x| = 0,$$
$$u(0,x) = \sin x.$$

Then

$$\Phi(q)=\begin{cases}0, & q=[-1,1],\\ +\infty, & q\prec -1,\ q\succ 1,\end{cases}$$

$$F(t,x,\xi)=\begin{cases}0, & \xi\in[x-t,x+t],\\ +\infty, & \xi\prec x-t,\ \xi\succ x+t,\end{cases}$$

and the viscosity solution can be represented by formula

$$u(t,x)=\min_{\xi\in[x-t,x+t]}\big[\sin(\xi)\big].$$

It is clear, that if we construct the Legrendre transform of hamiltonian (8), then we easy define the fundamental solution. Next we explain, how we can define the Legrendre transform, when $x\in R^n$, $n>1$.

Let $x\in R^2$. Then the Legrendre transform of

$$H(p_1,p_2)=\max_{i=1,\dots,m}\big((a_1^ip_1+a_2^ip_2)+b_i\big)$$

can be constructed in such way:

if $m=1$, then

$$\Phi(q)=\begin{cases}-b_i, & q=a^i,\\ +\infty, & q\neq a^i,\end{cases}$$

if $m=2$, then $\Phi(q)$ is defined in the parametric form

$$\begin{cases}\Phi(s)=(b_1-b_2)s-b_1,\\ q_1=(a_1^2-a_1^1)s+a_1^1,\\ q_2=(a_2^2-a_2^1)s+a_2^1,\end{cases}$$

where $s\in[0,1]$, in other points of R^2 the function $\Phi(q)=+\infty$,

if $m\geqslant 3$, then define $Q=co\{a^i\}$-convex hull of set $\{a^i,\ i=1,\dots,m\}$ and $Q_k=co\{a^{k_1},a^{k_2},a^{k_3}\}$, where $k_1,k_2,k_3\in\{1,\dots,m\}$, and $a^i\notin Q_k$, when $i\notin\{k_1,k_2,k_3\}$. Then $\Phi(q)=\max_k\{(\alpha_k,q)+\beta_k\}$, when $q\in Q$, and $\Phi(q)=+\infty$, if $q\notin Q$. The coefficients α_k,β_k are determined from the identity

$$\begin{vmatrix} q_1-a_1^{k_1} & q_2-a_2^{k_1} & (\alpha_k,q)+\beta_k+b_{k_1}\\ a_1^{k_2}-a_1^{k_1} & a_2^{k_2}-a_2^{k_1} & b_{k_1}-b_{k_2}\\ a_1^{k_3}-a_1^{k_1} & a_2^{k_3}-a_2^{k_1} & b_{k_1}-b_{k_3}\end{vmatrix}\equiv 0.$$

The similar structure of the Legrendre transform for the hamiltonians (7) may be realized in R^n.

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REZIUMĖ

Apie Hamiltono-Jakobi lygčių sprendinių išraiškas

G. Gudynas

Straipsnyje analizuojamos Hamiltono-Jakobi lygčių sprendinių išraiškos, kai hamiltonianas užduodamas kaip tiesinių funkcijų gaubiamoji.

Raktiniai žodžiai: Hamiltono-Jakobi lygtys, pusiautolydūs iš apačios sprendiniai, Hopfo formulė.