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Inverstigation of the spectrum for Sturm–Liouville problems with a nonlocal boundary condition

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Abstract. In this paper, we analyze the Sturm–Liouville problem with one classical first type boundary condition and the other Samarskii–Bitsadze type nonlocal boundary condition. We investigate how the spectrum of this problem depends on the parameters γ and ξ of the nonlocal boundary condition. Some new results are given as graphs of the characteristic function.

Keywords: Sturm–Liouville problem, two-point Nonlocal Boundary Condition, complex eigenvalues.

Introduction

Problems with Nonlocal Boundary Conditions (NBCs) are an area of the fast developing differential equations theory. Problems of this type arise in various fields of physics, biology, biotechnology, etc. Nonlocal conditions arise when we cannot measure data directly at the boundary. In this case, a problem is formulated where the value of the solution or a derivative is linked to a few points or the whole interval.

A. Samarskii and A. Bitsadze have formulated and investigated a new problem for uniformly elliptic equations. These boundary conditions connect values of the desired solution on the boundary with inner points of the domain [1]. J. Cannon investigated parabolic problem with integral NBCs [2]. A multi-point nonlocal boundary problem for the second-order ordinary differential equation was also investigated in [3, 4, 5]. An eigenvalue problem with the nonlocal condition is closely linked with a boundary problem for a differential equation with NBCs.

In this paper, we analyze an eigenvalue problem for a stationary differential operator with one classical and another two-point NBC. We investigate how the real and complex eigenvalues of this problem depend on the parameters γ and ξ of the NBC. As the theoretical investigation of the complex spectrum is a very difficult problem, we present the result of modelling and illustrate the existing situation in graphs.

1 Sturm–Liouville problem with NBC

Let us analyze the Sturm–Liouville problem with one classical boundary condition

$$-u'' = \lambda u, \quad t \in (0,1), \tag{1}$$
$$u(0) = 0, \tag{2}$$

and another nonlocal two-point boundary condition of Samarskii–Bitsadze type

$$u(\xi) = \gamma u(1-\xi), \tag{3}$$

with the parameters $\gamma \in \overline{\mathbb{R}}$ and $\xi \in [0,1]$.

Remark 1. We discuss the critical cases that will not be discussed later on.

Case $\gamma = 0$. If $\xi = 0$, then have problem (1), (2) with one boundary condition $u(0) = 0$ only. If $0 < \xi \leqslant 1$, then we have the classical boundary value problem in the interval $[0,\xi]$ with boundary conditions $u(0) = 0$, $u(\xi) = 0$, and its eigenvalues and eigenfunctions are

$$\lambda_k = \left(\frac{\pi k}{\xi}\right)^2, \qquad u_k(t) = \sin\left(\frac{\pi k t}{\xi}\right), \quad k \in \mathbb{N}. \tag{4}$$

In the case $0 < \xi < 1$ we also have the initial value problem in the interval $[\xi, 1]$ with the initial condition $u(\xi) = 0$ only.

Case $\gamma = \infty$. If $\xi = 1$, we have problem (1)–(2) with one boundary condition $u(0) = 0$. If $0 \leqslant \xi < 1$ then we have the same situation as in Case $\gamma = 0$ with the boundary value problem in the interval $[0, 1-\xi]$ and the initial value problem in the interval $[1-\xi, 1]$. In the case $\xi = 0$ we have the boundary value problem only.

Case $\xi = \frac{1}{2}$. If $\gamma = 1$, then we have problem (1)–(2) with one boundary condition $u(0) = 0$. If $\gamma \neq 1$, then we have the boundary value problem in the interval $[0, \frac{1}{2}]$ and the initial value problem in the interval $[\frac{1}{2}, 1]$ (see Case $\gamma = 0$).

Case $\xi = 0$ or $\xi = 1$ ($\gamma \neq 0$, $\gamma \neq \infty$). We have the same case as in Case $\gamma = 0$ ($\xi = 1$).

Let us return to the problem (1)–(3) and consider that $0 < \xi < 1$, $\xi \neq \frac{1}{2}$, $\gamma \neq 0$ and $\gamma \neq \infty$. If $\lambda = 0$, then the function $u(t) = ct$ satisfies problem (1)–(2). By substituting this solution into NBC, we derive that there exists a nontrivial solution ($c \neq 0$) if $c\xi = \gamma c(1-\xi)$. So, the following lemma is valid.

Lemma 1. *There exists the eigenvalue $\lambda = 0$ if and only if $\gamma = \frac{\xi}{1-\xi}$.*

In the general case, if $\lambda \neq 0$ and the eigenvalues $\lambda = q^2$, $q \in \mathbb{C}_q \setminus \{0\}$ (here $\mathbb{C}_q := \{q \in \mathbb{C}\colon \operatorname{Re} q > 0 \text{ or } \operatorname{Im} q \geqslant 0 \text{ in the case } \operatorname{Re} q = 0\}$), then the solution of problem (1)–(2) is $u(t) = c\sin(qt)$. The second boundary condition gives equation:

$$c\sin(q\xi) = c\gamma \sin\bigl(q(1-\xi)\bigr). \tag{5}$$

There exists a nontrivial solution, if q is the root of the equation

$$f(z) := \gamma \sin\bigl(q(1-\xi)\bigr) - \sin(q\xi) = 0. \tag{6}$$

So, if the next equation is valid

$$\sin(q\xi) = 0, \qquad \sin\bigl(q(1-\xi)\bigr) = 0, \tag{7}$$

then Eq. (6) is valid for all γ. In this case, we get *constant eigenvalues* (which do not depend on the parameter $\gamma \in \mathbb{R}$) $\lambda = {c_k}^2$, $k \in \mathbb{N}$ and (constant eigenvalue points) c_k are the roots of system (7). We extract the second equation of the system (7) and use the first equation:

$$0 = \sin\bigl(q(1-\xi)\bigr) = \sin q\,\cos(\xi q) - \cos q\,\sin(\xi q) = \pm\sin q.$$

We obtain a new system

$$\sin q = 0, \qquad \sin(q\xi) = 0, \tag{8}$$

the roots of which are constant eigenvalues $\lambda = {c_k}^2$ of problem (1)–(3). Such a system (8) is investigated in the article [6]: *if the parameter ξ is an irrational number, then constant eigenvalues do not exist.*

So, constant eigenvalues exist only for rational $\xi = r = \frac{m}{n} \in [0,1]$ and those eigenvalues are equal to $\lambda = c_k^2$, $c_k = \pi kn$, $k \in \mathbb{N}$. All nonconstant eigenvalues $\lambda = \lambda(q) = q^2(\gamma)$ are γ-points of the meromorphic function

$$\gamma(q) = \gamma_c(q) = \frac{\sin(q\xi)}{\sin\bigl(q(1-\xi)\bigr)}, \tag{9}$$

for $\xi \notin \mathbb{Q}$.

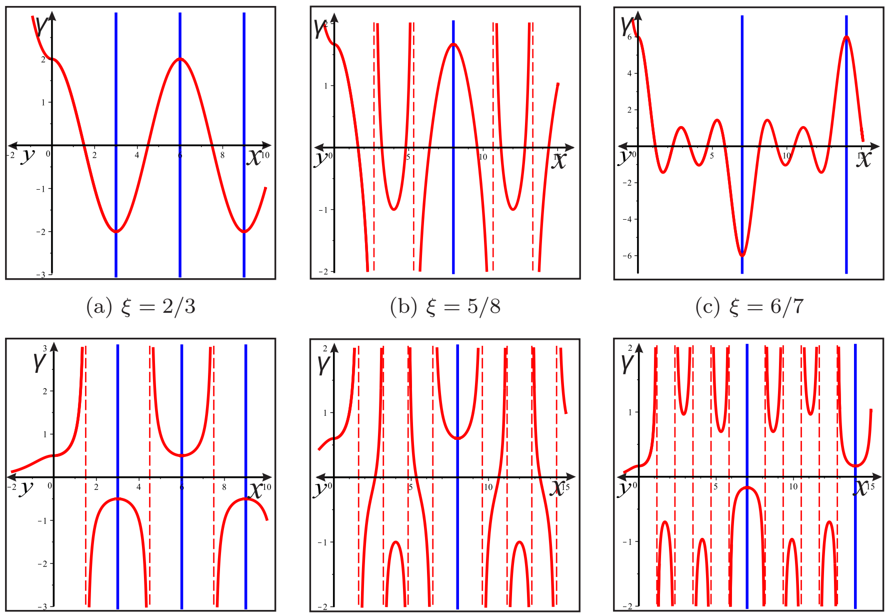
For the investigation of the constant eigenvalues as well as for the analysis of complex eigenvalues, zeroes and poles points of the characteristic function are important. The poles of the function (9) are eigenvalues of the problem (1)–(3) in the case $\gamma = \infty$.

All zeroes and poles of the meromorphic function $\gamma_c(q)$ lie on the positive side of the real axis. Zero points z_k of the function $\gamma_c(q)$ are of the first order. These positive zeroes are equal to $z_k = \pi k/\xi$, $k \in \mathbb{N}$. Pole points are equal to $p_k = \pi k/(1-\xi)$, $k \in \mathbb{N}$ and they are of the first order. If these pole points are coincide with zeroes at the points $c_k = \pi kn$, we have constant eigenvalue points.

Remark 2. We investigated the problem (1)–(3) with the parameter $\xi > 1/2$, because, if we have $\xi < 1/2$, then $\bar{\xi} = 1-\xi > 1/2$, and $u(\xi) = \gamma u(1-\xi)$ is equivalent to $u(\bar{\xi}) = 1/\bar{\gamma}u(1-\bar{\xi})$, where $\bar{\gamma} = 1/\gamma$. Then poles and zeroes are equal to $z_k = \pi k/(1-\bar{\xi})$, $p_k = \pi k/\bar{\xi}$, (i.e. the points of zeroes become points of poles and points of poles become points of zeroes). For example, in Fig. 1(a)–(b) the parameter ξ is equal to $\xi > 1/2$ and in the pictures (d)–(f) it is $\xi < 1/2$. Thus, it is enough to investigate problem (1)–(3) with the parameter $\xi > 1/2$.

1.1 Real and complex eigenvalues

We take q only in the rays $q = x \geqslant 0$ and $q = -\imath x$, $x \leqslant 0$ instead of $q \in \mathbb{C}_q$. We obtain positive eigenvalues in the case $x > 0$ and we have negative eigenvalues in the



(a) $\xi = 2/3$ (b) $\xi = 5/8$ (c) $\xi = 6/7$

(d) $\xi = 1/3$, $\bar{\xi} = 2/3$ (e) $\xi = 3/8$, $\bar{\xi} = 5/8$ (f) $\xi = 1/7$, $\bar{\xi} = 6/7$

Fig. 1. Real characteristic functions $\gamma(\pi q)$ for various ξ.

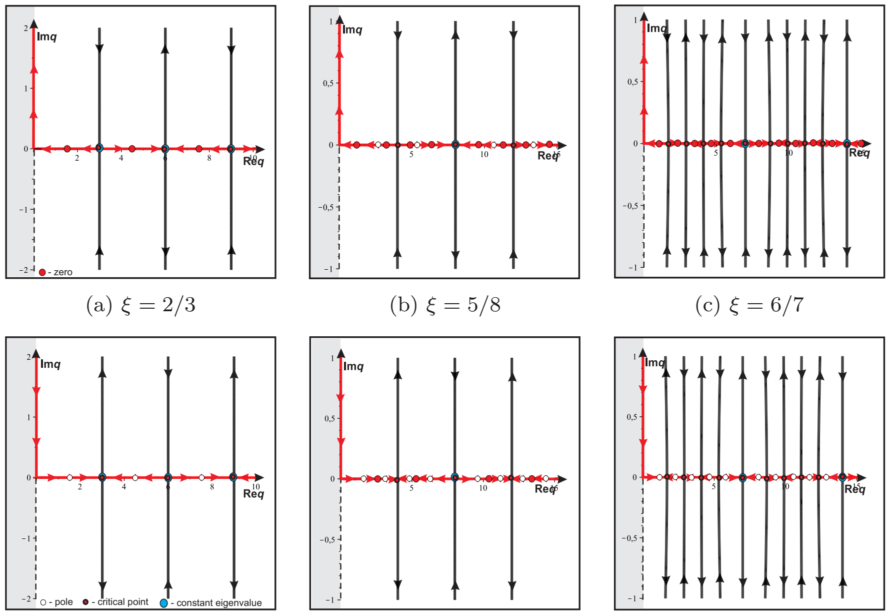
case $x < 0$. The point $x = 0$ corresponds to $\lambda = 0$. So, for complex function (9), we can write the *real characteristic function* [6]:

$$\gamma(x) := \begin{cases} \frac{\sinh(x\xi)}{\sinh(x(1-\xi))}, & x < 0, \\ \frac{\sin(x\xi)}{\sin(x(1-\xi))}, & x \geqslant 0. \end{cases} \tag{10}$$

The graphs of this real characteristic function $\gamma(x)$ for some parameter ξ values are presented in Fig. 1 (the x-axis is scaled π times and $x = 1$ is really $x = \pi$ in all the figures). This function $\gamma(x)$ is useful for the investigation of zero. The vertical blue (solid) lines correspond to constant eigenvalues, vertical red (dashed) lines cross the x-axis at the points of poles. If $\xi < 1/2$ and $0 < \gamma < \frac{\xi}{1-\xi}$ (Fig. 1(d)–(f)) or if $\xi > 1/2$ and $\gamma > \frac{\xi}{1-\xi}$ (Fig. 1(a)–(b)), then and only then there exists negative eigenvalue. In Fig. 1, when the parameter ξ is rational, we get constant eigenvalue points.

Let us investigate problem (1)–(3), with the parameter $\gamma \in \mathbb{R}$. The restriction $\operatorname{Im}\gamma(q) = 0$, of the complex characteristic function (9), is called the *complex–real characteristic function*. The restricted function $\gamma(q) : \mathcal{N} \to \mathbb{R}$ is defined on some subset: $\mathcal{N} := \gamma^{-1}(\mathbb{R}) := \{q \in \mathbb{C}_q\colon \operatorname{Im}\gamma(q) = 0\}$. In the general case, the subset is a union of the curves in the domain $\mathbb{C}_q$. We call the point $q_c \in \mathbb{C}_q$, such that $\gamma'(q_c) = 0$, a critical point. All the critical points of problem (1)–(3) are real.

In Fig. 2 it is shown, how the domain $\mathcal{N}$ is changing the value of the parameter ξ. When we are increasing the value of parameter ξ, the poles are moving to the right, and the zeroes are moving to the left. When zeroes and poles coincide, we have a points of the constant eigenvalue. In Fig. 2, we can see that points of the constant eigenvalues are also identical with the critical points.



(a) $\xi = 2/3$ (b) $\xi = 5/8$ (c) $\xi = 6/7$

(d) $\xi = 1/3$, $\bar{\xi} = 2/3$ (e) $\xi = 3/8$, $\bar{\xi} = 5/8$ (f) $\xi = 1/7$, $\bar{\xi} = 6/7$

Fig. 2. Domain $\mathcal{N}$ with various ξ for the complex-real function $\gamma(\pi q)$.

2 Conclusions

In this paper, the spectrum of the Sturm–Liouville problem with the classical boundary condition on the left side of the interval and the Samarskii–Bitsadze type nonlocal boundary condition has been investigated.

- The spectrum of problem (1)–(3) has no constant eigenvalues for irrational ξ. There are countably many nonconstant and constant eigenvalues for rational ξ. All the constant eigenvalues are real positive numbers.
- If $\xi < 1/2$ and $0 < \gamma < \frac{\xi}{1-\xi}$ or if $\xi > 1/2$ and $\gamma > \frac{\xi}{1-\xi}$, then and only then there exists negative eigenvalue. In case $\gamma = \frac{\xi}{1-\xi}$, we have the eigenvalue $\lambda = 0$.

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REZIUMĖ

Šturmo ir Liuvilio uždavinio su integraline nelokaliąja sąlyga spektro tyrimas

A. Skučaitė, A. Štikonas

Straipsnyje pateikiami nauji rezultatai, gauti tiriant diferencialinį Šturmo ir Liuvilio uždavinį su viena nelokaliąja integraline kraštine sąlyga spektrą. Ištirta spektro priklausomybė nuo nelokaliųjų sąlygų parametrų γ ir ξ nuo integralo simetriniame intervale. Daugelis rezultatų pateikiama charakteristinių funkcijų grafikais.

Raktiniai žodžiai: Šturmo ir Liuvilio uždavinys, nelokaliosios kraštinės sąlygos, kompleksinės tikrinės reikšmės.