LIETUVOS MATEMATIKOS RINKINYS. LMD DARBAI
52 tomas, 2011, 297–302

ISSN 0132-2818
www.mii.lt/LMR/

# Investigation of the Sturm–Liouville problems with integral boundary condition

**Agnė Skučaitė[1], Artūras Štikonas[1,2]**

[1] *Institute of Mathematics and Informatics, Vilnius University*
Akademijos 4, Vilnius, LT-08663
[2] *Faculty of Mathematics and Informatics, Vilnius University*
Naugarduko 24, LT-03225 Vilnius
E-mail: agne.skucaite@mii.vu.lt; arturas.stikonas@mif.vu.lt

**Abstract.** This paper presents some new results on the spectrum of a complex plane for the second order Finite-Difference Scheme with one integral type nonlocal boundary condition (NBC). We analyze how complex eigenvalues of these problems depend on the parameters of the integral NBC. The integral conditions are approximated by the trapezoidal rule or by Simpson's rule.

**Keywords:** Finite-Difference Scheme, Nonlocal Boundary Condition, complex eigenvalues.

## Introduction

Problems with integral *Nonlocal Boundary Condition* (NBC) arise in various fields of mathematical physics, biology, biotechnology etc. J. Cannon investigates *Boundary Value Problem* (BVP) with integral type NBC [1]. Complex eigenvalues with NBC are less investigated than the real cases. Some results of this problem about complex eigenvalues are published in [3, 4].

## 1 A Sturm–Liouville problem with an integral NBC

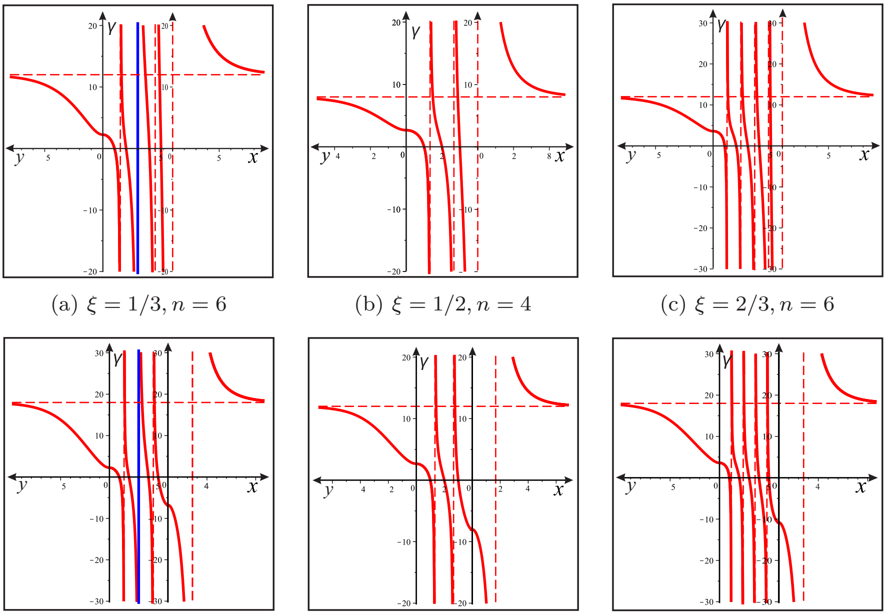
Let us consider a Sturm–Liouville problem with one classical boundary condition

$$-u'' = \lambda u, \quad t \in (0,1), \quad u(0) = 0, \tag{1}$$

and an integral NBC:

$$u(1) = \gamma \int_{\xi}^{1} u(t)\,dt \quad \text{or} \quad u(1) = \gamma \int_{0}^{\xi} u(t)\,dt, \tag{$2_{1,2}$}$$

with the parameters $\gamma \in \mathbb{R}$ and $\xi \in [0,1]$. Note that, if we write the index in the formula, as in $(2_{1,2})$, then first part of the formula is related to Case 1 and the second part is related to Case 2. If we use one index, then the formula is related to the one case. In Case 1 for $\xi = 0$ and Case 2 for $\xi = 1$, we have the same integral NBC. For $\gamma = 0$ or $\xi = 0$ in Case 1 and $\gamma = 0$ or $\xi = 1$ in Case 2, we have the problem with



(a) $\xi = 1/3, n = 6$ (b) $\xi = 1/2, n = 4$ (c) $\xi = 2/3, n = 6$

(d) $\xi = 1/3, n = 6$ (e) $\xi = 1/2, n = 4$ (f) $\xi = 2/3, n = 6$

**Fig. 1.** Generalized $\mathbb{R}$ characteristic functions for various $\xi$ of discrete problems in Case 1: (a)–(c) approximated by trapezoid; (d)–(f) by Simpson's rules.

classical boundary conditions and their eigenvalues and eigenfunctions of them are well-known [2]:

$$\lambda_k^0 = \left(\pi q_k^0\right)^2, \qquad u^{k;0}(t) = \sin\left(\pi q_k^0 t\right), \quad q_k^0 = k \in \mathbb{N} := \{1, 2, 3, \ldots\}. \tag{3}$$

The characteristic function and its domain $\mathcal{N}$ for these problems are described in [4].

## 2 The case of an approximation by the trapezoidal rule

In the interval $[0, 1]$, we introduce a uniform grid $\omega^h = \{t_j = jh,\ j = \overline{0, n};\ n \in \mathbb{N},\ nh = 1\}$. Also, we make an assumption, that $\xi$ is coincident with a grid point, i.e., $\xi = mh = m/n$, $m = \overline{0, n}$. Let us denote the greatest common divisor by $K := \gcd(n, m)$ and $N := n/K$, $M := m/K$. Then $\xi = M/N$, too. We approximate differential problem (1), ($2_{1,2}$) by the *Finite-Difference Scheme* (FDS):

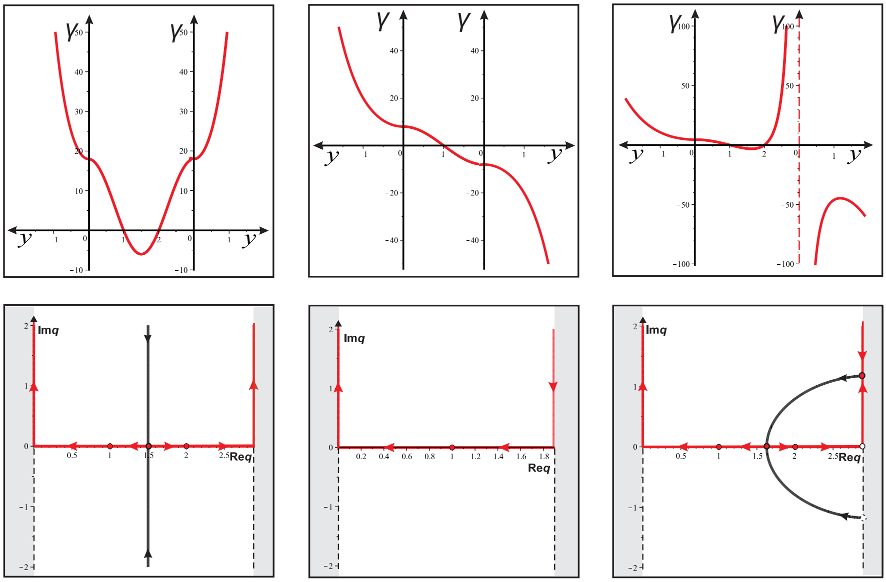
$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = \overline{1, n-1}, \qquad U_0 = 0, \tag{4}$$

$$U_n = \gamma h\left(\frac{U_m + U_n}{2} + \sum_{k=m+1}^{n-1} U_k\right), \qquad U_n = \gamma h\left(\frac{U_0 + U_m}{2} + \sum_{k=1}^{m-1} U_k\right). \tag{$4_{1,2}$}$$

We investigate eigenvalues for the FDS. Rewrite Eq. (4) in another form:

$$U_{j+1} - 2\cos(\pi qh)U_j + U_{j-1} = 0, \qquad \lambda = \tfrac{4}{h^2}\sin^2(\pi qh/2), \qquad U_0 = 0, \tag{5}$$

where $q = x + iy \in \mathbb{C}_q^h = \{q\colon 0 < x < n\} \cup \{q\colon x = 0,\ y \geqslant 0\} \cup \{q\colon x = n, y \geqslant 0\}$. We note that, for differential problem (1)–(2) eigenvalues are defined by the formula $\lambda = q^2$, $q \in \mathbb{C}_q = \{q\colon 0 < x\} \cup \{q\colon x = 0,\ y \geqslant 0\}$ [4].



(a) Case 2, $\xi = 1/3$, $n = 3$ (b) Case 2, $\xi = 1/2$, $n = 2$ (c) Case 2, $\xi = 2/3$, $n = 3$

**Fig. 2.** Generalized $\mathbb{R}$ and domain $\mathcal{N}$ of $\mathbb{C}$-$\mathbb{R}$ characteristic functions for problem (4)–($4_2$).

The general solution of difference equation (5) is $U = C_1 \sin(\pi q t_j) + C_2 \cos(\pi q t_j)$ for $q \neq 0, n$; $U = C_1 t_j + C_2$ for $q = 0$; $U = C_1(-1)^j t_j + C_2(-1)^j$ for $q = n$.

We have the eigenvalue $\lambda = 0$ for problem (4)–($4_{1,2}$), if and only if $\gamma = \frac{2}{1-\xi^2}$ in Case 1 and $\gamma = \frac{2}{\xi^2}$ in Case 2 (the same conditions are for the differential case). We find the eigenvalue $\lambda = 4/h^2$, if and only if $\gamma = \frac{2}{h^2} \cdot \frac{2}{1-(-1)^{n-m}}$ in Case 1 ($n - m$ is odd), and $\gamma = \frac{2}{h^2} \cdot \frac{2(-1)^n}{(-1)^m - 1}$ in Case 2 ($m$ is odd).

If $\gamma = 0$, we have the classical BCs and all the $n-1$ eigenvalues for the classical FDS are positive and algebraically simple and do not depend on the parameter $\xi$:

$$\lambda_k^0 = \frac{4}{h^2} \sin^2\left(\pi q_k^0 h/2\right), \qquad U_j^{k;0} = \sin\left(\pi q_k^0 t_j\right), \quad q_k^0 = k = \overline{1, n-1}. \tag{6}$$

If $q = x \in (0, 1/h)$, then $\lambda \in (0, 4/h^2)$ and we calculate the eigenvalues of problem (4)–($4_{1,2}$) by the formula $\lambda_k = \frac{4}{h^2} \sin^2(\frac{\pi x_k h}{2})$, where $x_k$ are roots of the equation:

$$\sin(\pi x) - \gamma h \tan^{-1}(\pi x h/2)\big(\cos(\xi \pi x) - \cos(\pi x)\big)/2 = 0, \tag{$7_1$}$$

$$\sin(\pi x) - \gamma h \tan^{-1}(\pi x h/2) \sin^2(\xi \pi x/2) = 0. \tag{$7_2$}$$

The constant eigenvalue points are equal to:

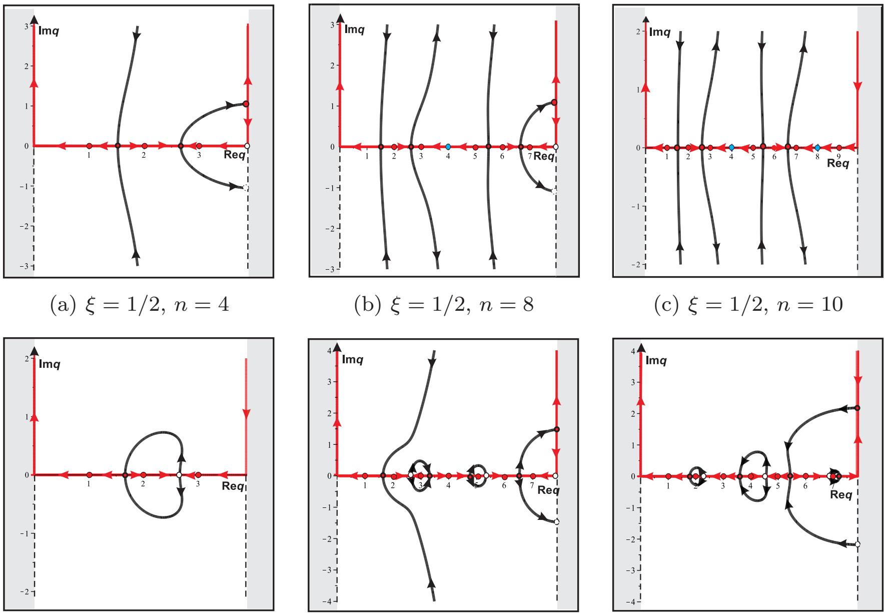
$$c_k = Nk, \quad \text{for odd } M, \qquad c_k = 2Nk, \quad \text{for even } M, \tag{$8_1$}$$

$$c_k = Nk, \quad \text{for odd } N - M, \qquad c_k = 2Nk, \quad \text{for even } N - M. \tag{$8_2$}$$

$k \in \mathbb{N}$ such that $c_k \in (0, n)$.

Other (nonconstant) eigenvalues (which depend on the parameter $\gamma$) as $\gamma$-values of the $\mathbb{C}$–$\mathbb{R}$ characteristic function (see Figs. 1(a)–1(c) and 2) are defined on the set $\mathbb{C}_q^h$:

$$\gamma = \frac{\sin(\pi q)}{\cos(\xi \pi q) - \cos(\pi q)} \cdot \tan\frac{\pi q h}{2} \cdot \frac{2}{h}, \qquad \gamma = \frac{\sin(\pi q)}{2\sin^2(\xi \pi q/2)} \cdot \tan\frac{\pi q h}{2} \cdot \frac{2}{h}. \tag{$9_{1,2}$}$$




(a) $\xi = 1/2$, $n = 4$ (b) $\xi = 1/2$, $n = 8$ (c) $\xi = 1/2$, $n = 10$

(d) $\xi = 3/4$, $n = 4$ (e) $\xi = 3/4$, $n = 8$ (f) $\xi = 7/8$, $n = 8$

**Fig. 3.** Projections of the $\mathbb{C}$-$\mathbb{R}$ characteristic functions (domain $\mathcal{N}$) in Case 2.

If $hq$ is a sufficiently small number, then $\tan\frac{\pi qh}{2} \cdot \frac{2}{\pi qh} \approx 1$. It follows that, in this case, the discrete characteristic function is similar to the characteristic function of the differential problem [2].

All the eigenvalues in Case 1 are real, complex eigenvalues do not exist for any value of NBC parameters $n$ and $m$ as in the differential case. For some values of $n$ and $m$, the complex part spectrum does not exist in Case 2, too (see Fig. 2). In Fig. 3, we see how complex part of the spectrum depends on the FDS parameter $n$ (number of the grid points). The grid point $q = n$ is a pole point for even $n - m$ in Case 1 and even $m$ in Case 2.

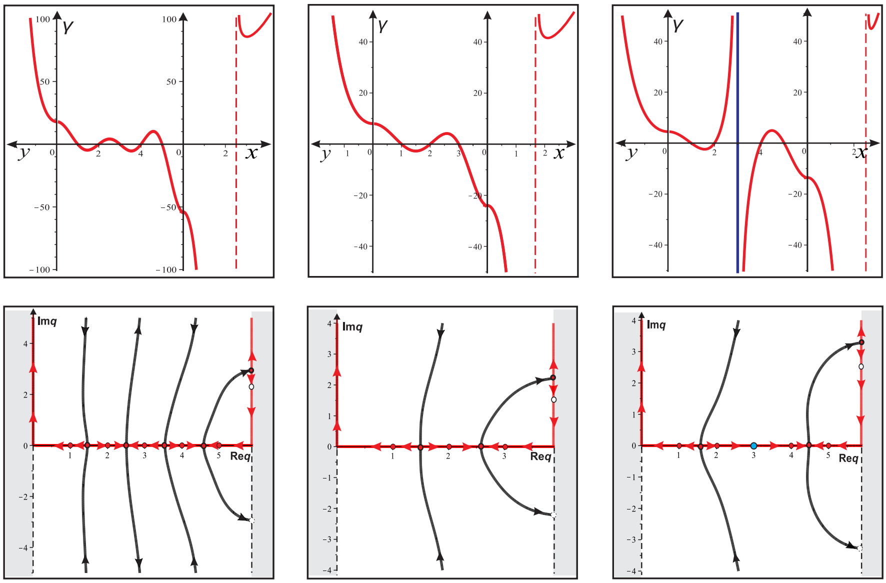
## 3 The case of an approximation by Simpson's rule

In the interval $[0,1]$, we introduce a uniform grid $\omega^h = \{t_j = jh,\ j = \overline{0,2n};\ 2nh = 1\}$. Also, we make the an assumption, that $\xi$ is coincident with the grid point, i.e., $\xi = 2mh = m/n$, $m = \overline{0,n}$. We approximate differential problem (1), $(2_{1,2})$ by FDS:

$$\frac{U_{j-1} - 2U_j + U_{j+1}}{h^2} + \lambda U_j = 0, \quad j = \overline{1, 2n-1}, \qquad U_0 = 0, \tag{10}$$

$$U_{2n} = \frac{\gamma h}{3}\left(U_{2m} + U_{2n} + 4\sum_{k=m+1}^{n} U_{2k-1} + 2\sum_{k=m+1}^{n-1} U_{2k}\right), \tag{$10_1$}$$

$$U_{2n} = \frac{\gamma h}{3}\left(U_0 + U_{2m} + 4\sum_{k=1}^{m} U_{2k-1} + 2\sum_{k=1}^{m-1} U_{2k}\right). \tag{$10_2$}$$



(a) $\xi = 1/3, n = 6$ (b) $\xi = 1/2, n = 4$ (c) $\xi = 2/3, n = 6$

**Fig. 4.** Generalized $\mathbb{R}$ and $\mathbb{C}$-$\mathbb{R}$ characteristic functions for various $\xi$ for discrete problem (10), $(10_2)$.

Let us denote the greatest common divisor $K := \gcd(2n, 2m)$ and $N := 2n/K$, $M := 2m/K$. Then $\xi = M/N$, too. We can rewrite Eq. (10) in form (5) as for FDS (4).

The boundary condition $U_0 = 0$ yields that $C_2 = 0$. By substituting such a solution function into NBC (10), we derive that the eigenvalue exists if $q = q(\gamma, \xi)$ is the root of the equation for $q \neq 0, n$:

$$\sin(\pi q) - \frac{\gamma h}{3}\big(\cos(\xi\pi q) - \cos(\pi q)\big) \cdot \frac{3\sin(\pi q h)}{(2 + \cos(\pi q h))} = 0, \qquad (11_1)$$

$$\sin(\pi q) - \frac{2\gamma h}{3}\sin^2(\xi\pi q/2) \cdot \frac{3\sin(\pi q h)}{(2 + \cos(\pi q h))} = 0. \qquad (11_2)$$

We obtain the eigenvalue $\lambda = 0$ for problem (10)–$(10_{1,2})$, if and only if $\gamma = \frac{2}{1-\xi^2}$ in Case 1 and $\gamma = \frac{2}{\xi^2}$ in Case 2 (the same conditions are for differential (1)–(2) and for FDS (4)–$(4_{1,2})$).

If $q = x \in (0, 1/h)$, then $\lambda \in (0, 4/h^2)$ and we calculate the eigenvalues of problem (10)–$(10_{1,2})$ by the formula $\lambda_k = \frac{4}{h^2}\sin^2(\frac{\pi x_k h}{2})$, where $x_k$ are roots of the equation:

$$\sin(\pi x) - \frac{\gamma h}{3}\big(\cos(\xi\pi x) - \cos(\pi x)\big) \cdot \frac{2 + \cos(\pi x h)}{\sin(\pi x h)} = 0, \qquad (12_1)$$

$$\sin(\pi x) - \frac{2\gamma h}{3}\sin^2\left(\frac{\xi\pi x}{2}\right) \cdot \frac{2 + \cos(\pi x)h}{\sin(\pi x h)} = 0. \qquad (12_2)$$

Constant eigenvalue points $c_k$ are describes by formula (8) (the same formula is for (4)–$(4_{1,2})$). Other (nonconstant) eigenvalues (which depend on the parameter $\gamma$) as

$\gamma$-values of the $\mathbb{C}$–$\mathbb{R}$ characteristic function (see Figs. 1(d)–1(f) and 4) are defined on the set $\mathbb{C}_q^h$ [4]:

$$\gamma = \frac{\sin(\pi q)}{\cos(\xi\pi q) - \cos(\pi q)} \cdot \frac{3\sin(\pi q h)}{2+\cos(\pi q h)}, \qquad \gamma = \frac{\sin(\pi q)}{2\sin^2(\xi\pi q/2)} \cdot \frac{3\sin(\pi q h)}{2+\cos(\pi q h)}. \tag{$13_{1,2}$}$$

In Case 1, if nonlocal boundary condition ($2_1$) is approximated by Simpson's rule, complex eigenvalues do not exist as in the differential case. In Case 2, the spectrum of complex eigenvalues is more complicated (see Fig. 4). In this case, the complex part of the spectrum depends on the FDS parameter $n$ (number of the grid points). We see that, in the case of FDS (10)–($10_{1,2}$), there exists one pole point $q = n + iy$, where $\cosh(\pi y h) = 2$. Also, some of the complex eigenvalue curves make loops.

## 4 Conclusions

The spectra of FDS's (4)–($4_{1,2}$) and (10)–($10_{1,2}$) in Cases 1 and 2 are different. Like in the case of the differential problem, in Case 1 of FDSs (4)–($4_1$) and (10)–($10_1$) only real eigenvalues exist. With an increase in the value $n$, the spectra of FDSs (4)–($4_{1,2}$) and (10)–($10_{1,2}$) are become more similar to that spectrum of the differential problem. In Case 2 of both FDSs (4), ($4_2$) and (10), ($10_2$), complex part of the spectrum is very complicated for some values of NBCs parameters $\gamma$, $\xi$ and $n$.

## References


[1] J.R. Cannon. The solution of the heat equation subject to specification of energy. *Quart. Appl. Math.*, **21**(2):155–160, 1963.

[2] S. Pečiulytė, O. Štikonienė and A. Štikonas. Sturm–Liouville problem for stationary differential operator with nonlocal integral boundary condition. *Math. Model. Anal.*, **10**(4):377–392, 2005.

[3] A. Skučaitė, K. Skučaitė-Bingelė, S. Pečiulytė and A. Štikonas. Investigation of the spectrum for the Sturm–Liouville problem with one integral boundary condition. *Nonlinear Anal., Model. Control*, **15**(4):501–512, 2010.

[4] A. Štikonas and O. Štikonienė. Characteristic functions for Sturm–Liouville problems with nonlocal boundary conditions. *Math. Model. Anal.*, **14**(2):229–246, 2009.



REZIUMĖ

**Šturmo ir Liuvilio uždavinio su nelokaliosiomis integralinėmis sąlygomis tyrimas**
*A. Skučaitė, A. Štikonas*

Straipsnyje pateikiami nauji rezultatai, gauti tiriant antros eilės baigtinių skirtumų schemos su viena integraline nelokaliąja sąlyga kompleksinės spektro dalies struktūrą. Ištirta kompleksinių tikrinių reikšmių priklausomybė nuo nelokaliųjų sąlygų parametrų. Integralinės sąlygos aproksimuojamos dviem būdais: trapecijų arba Simpsono formule.


*Raktiniai žodžiai*: baigtinių skirtumų schema, nelokaliosios kraštinės sąlygos, kompleksinės tikrinės reikšmės.