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# Distribution of the critical and other points in boundary problems with nonlocal boundary condition

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**Abstract.** In this paper the Sturm–Liouville problem with one classical and other nonlocal two-point or integral boundary condition is investigated. There are critical points of the characteristic function analysed. We investigate how distribution of the critical points depends on nonlocal boundary condition parameters.

*Keywords:* Sturm–Liouville problem, nonlocal boundary condition.

## 1. Introduction

In recent scientific literature, a great attention is paid to differential problems with nonlocal boundary conditions. They are investigated both in foreign and in Lithuanian scientists papers. Differential problems with nonlocal two-point boundary conditions are investigated by A.V. Gulin, V.A. Morozova [1], V.A. Ilyin, E.I. Moiseev [2], N.I. Ionkin, E.A. Valikova [3], M. Sapagovas, A. Štikonas [7,8], S. Pečiulytė [4–6].

In this paper the Sturm–Liouville problem with one classical and other nonlocal two-point or integral boundary conditions is analysed. Problems with such boundary conditions were investigated in papers [4–6]. Dependence of such problems spectrums on nonlocal boundary conditions parameter $\gamma$ (parameter $\xi$ was fixed) was investigated in previous research. Furthermore, conditions, when constant, negative and only real eigenvalues exist were drawn in the articles.

In this paper we investigate critical points of real characteristic function. New results on constant and critical points distribution dependence on parameter $\xi$ are presented.

## 2. Problems with nonlocal boundary conditions

Let us analyze the Sturm–Liouville problem with one classical boundary condition

$$-u'' = \lambda u, \qquad t \in (0,1), \tag{1}$$

$$u(0) = 0, \tag{2}$$

and another nonlocal two-point boundary condition of Samarskii–Bitsadze or integral type:

$$u'(1) = \gamma u(\xi), \qquad \text{(Case 1)} \qquad (3_1)$$

$$u'(1) = \gamma u'(\xi), \qquad \text{(Case 2)} \qquad (3_2)$$

$$u(1) = \gamma u'(\xi), \qquad \text{(Case 3)} \qquad (3_3)$$

$$u(1) = \gamma u(\xi), \qquad \text{(Case 4)} \tag{$3_4$}$$

$$u(1) = \gamma \int_0^{\xi} u(t)\,\mathrm{d}t, \qquad \text{(Case 5)} \tag{$3_5$}$$

$$u(1) = \gamma \int_{\xi}^{1} u(t)\,\mathrm{d}t, \qquad \text{(Case 6)} \tag{$3_6$}$$

with the parameters $\gamma \in \mathbb{R}$ and $\xi \in [0, 1]$. Also we analyze the Sturm–Liouville problem (1) with boundary condition

$$u'(0) = 0 \tag{4}$$

on the left side and with nonlocal boundary conditions (3) on the right side of the interval. We enumerate these cases from Case 1′ till Case 6′ accordingly. Note that the index in references denotes the case. If there are no index then the rule (or results) holds on in all cases of nonlocal boundary conditions.

Let us define a *constant eigenvalue* as the eigenvalue $\lambda = q^2$ which does not depend on the parameter $\gamma \in \mathbb{C}$. For such a constant eigenvalue we define the *constant eigenvalue point* $q \in \mathbb{C}_q := \{z \in \mathbb{C}\colon -\pi/2 < \arg z \leqslant \pi/2 \text{ or } z = 0\}$ and the *constant eigenvalue $\gamma$-value point* $(q, \gamma) \in \mathbb{C}_q \times \mathbb{C}$, respectively. Other eigenvalues will be named as *nonconstant*.

All nonconstant eigenvalues (which depend on the parameter $\gamma$) are $\gamma$-points of the meromorphic functions $\gamma_c\colon \mathbb{C}_q \to \overline{\mathbb{C}}$. We call this function $\gamma_c$ as a *complex characteristic function*.

We call the point $q_c \in \mathbb{C}_q$, $q_c \neq 0$ such that $\gamma_c'(q_c) = 0$ a *critical point* for the complex characteristic function, and we call an image of the critical point $\gamma_c(q_c)$ *a critical value* of the complex characteristic function.

## 3. Real characteristic function

If we take $q$ only in the rays $q = x \geqslant 0$, $q = -\mathrm{i}x$, $x \leqslant 0$ instead of $q \in \mathbb{C}_q$, we get positive eigenvalues in case the ray $q = x > 0$, and we get negative eigenvalues in the ray $q = -x$, $x < 0$. The point $q = x = 0$ corresponds to $\lambda = 0$. We have two restrictions of the function $\gamma_c\colon \mathbb{C}_q \to \mathbb{R}$ on those rays: $\gamma_+(x) := \gamma_c(x + \mathrm{i}0)$ for $x \geqslant 0$ and $\gamma_-(x) := \gamma_c(0 - \mathrm{i}x)$ for $x \leqslant 0$. The function $\gamma_+$ corresponds to the case of positive eigenvalues, while the function $\gamma_-$ to that of negative eigenvalues. All the real eigenvalues

$$\lambda_k = \begin{cases} x_k^2, & \text{for } x_k \geqslant 0, \\ -x_k^2, & \text{for } x_k \leqslant 0, \end{cases} \qquad k \in \mathbb{N}, \tag{5}$$

can be investigated using a *real characteristic function* $\gamma\colon \mathbb{R} \to \mathbb{R}$:

$$\gamma(x) = \begin{cases} \gamma_+(x) = \gamma_c(x), & \text{for } x \geqslant 0, \\ \gamma_-(x) = \gamma_c(-\mathrm{i}x), & \text{for } x \leqslant 0. \end{cases}$$

We enumerate the eigenvalues using the classical case $\gamma = 0$. Eigenvalues $\lambda_k$ (and eigenvalue points $x_k$) depend on the parameter $\gamma$ continuously. Let us enumerate all the poles $p_k$, $k \in \mathbb{N}$ in an increasing order. Then the real characteristic function $\gamma$ is defined for $x \in \overline{P}_i := (p_{i-1}, p_i)$, $i \in \mathbb{N}$ and $\overline{P}_0 := (-\infty, 0)$. If $p_i$ is a constant eigenvalue point $c_j$ or $c_0$, then we add this point to the interval, i.e. $\overline{P}_i := (p_{i-1}, c_j]$ or $\overline{P}_i := [c_{j-1}, p_i)$ or $\overline{P}_i := [c_{j-1}, c_j]$. The point $x_{cr} \in \overline{P}_i$ is a *critical point* of real characteristic function, if $\gamma'(x_{cr}; \xi) = 0$. Note, that $p_k = p_k(\xi)$, $c_j = c_j(\xi)$, $x_{cr} = x_{cr}(\xi)$. For such a critical point $x_{cr}$ we define the *critical point* $(x_{cr}(\xi), \xi) \in \mathbb{R} \times [0, 1] \subset \mathbb{R}_x \times \mathbb{R}_\xi$. If the critical point is an extremum, i.e. the maximum or minimum point, then we use notation "extremum" instead "critical". Note, that the property "to be critical point" or "to be extremum" is only in $x$-direction.

We write expressions of the characteristic function in Cases 4, 1′ and 6′:

$$\gamma_4(x;\xi) := \begin{cases} \gamma_{4-}(x;\xi) = \dfrac{\sinh x}{\sinh(\xi x)}, & x \leqslant 0, \\ \gamma_{4+}(x;\xi) = \dfrac{\sin x}{\sin(\xi x)}, & x \geqslant 0; \end{cases} \tag{$6_4$}$$

$$\gamma_{1'}(x;\xi) := \begin{cases} \gamma_{1'-}(x;\xi) = \dfrac{x \sinh x}{\cosh(\xi x)}, & x \leqslant 0, \\ \gamma_{1'+}(x;\xi) = -\dfrac{x \sin x}{\cos(\xi x)}, & x \geqslant 0; \end{cases} \tag{$6_{1'}$}$$

$$\gamma_{6'}(x;\xi) := \begin{cases} \gamma_{6'-}(x;\xi) = \dfrac{x \cosh x}{2\cosh\left((1+\xi)\frac{x}{2}\right)\sinh\left((1-\xi)\frac{x}{2}\right)}, & x \leqslant 0, \\ \gamma_{6'+}(x;\xi) = \dfrac{x \cos x}{2\cos\left((1+\xi)\frac{x}{2}\right)\sin\left((1-\xi)\frac{x}{2}\right)}, & x \geqslant 0. \end{cases} \tag{$6_{6'}$}$$

In other cases of nonlocal boundary conditions characteristic functions and their graphs are presented in [4,7]. Characteristic functions coincide in Cases 1 and 5′, in Cases 2 and 4′, in Cases 4 and 2′ accordingly.

The spectrum of Sturm–Liouville problems (1)–(3) were investigated in papers [5–8]. There are presented lemmas on existence of characteristic functions zeroes, poles, minimums and maximums and conditions when constant eigenvalues exist. We noticed (see, [6]) that two negative real eigenvalues can exist in negative part of the real spectrum in problems (1)–(2), ($3_3$) and (1),(4), ($3_1$) for some $\gamma$ and $\xi$ values. Negative multiple and complex eigenvalues can also exist. In other cases of nonlocal boundary conditions one negative real eigenvalue exists for particular values of the parameter $\gamma$.

## 4. Critical points

Critical points of the characteristic function are important for investigation of multiple eigenvalues. Generalized eigenfunctions can exist for these points. In papers [5–7] critical points of the characteristic function were investigated in environment of a point, when values of parameter $\xi$ were fixed. In this paper, we present the interesting distribution of curves, that shows the change of the critical points (minimum and maximum points) of the characteristic function when $\xi \in [0, 1]$.

There are presented critical points distribution in some cases in Figs. 1–3 as an example. Note, that $x$-axe is scaled $\pi$ times and $x = 1$ is really $x = \pi$ in all figures. There are shown curves of critical points, zeroes (vertical line), poles (hyperbolas) points.

We analyze Sturm–Liouville problem (1)–(2), ($3_3$) (see, Fig. 1) as an example in detail. The graph of characteristic function minimums is also shown when $x < 0$, as in this problem two negative eigenvalues can exist in some cases. When $\xi = \frac{3}{4}$, we have one negative minimum of characteristic function (see Fig. 1, point 1), then pole (point 2), zero (point 3) and positive maximum (point 4) of characteristic function. In the crossing points of vertical lines and hyperbolas, we have constant eigenvalue point (point 6) which is turning type bifurcation point for critical points. Critical points curves intersect constant eigenvalue points. We noticed that minimum point appear

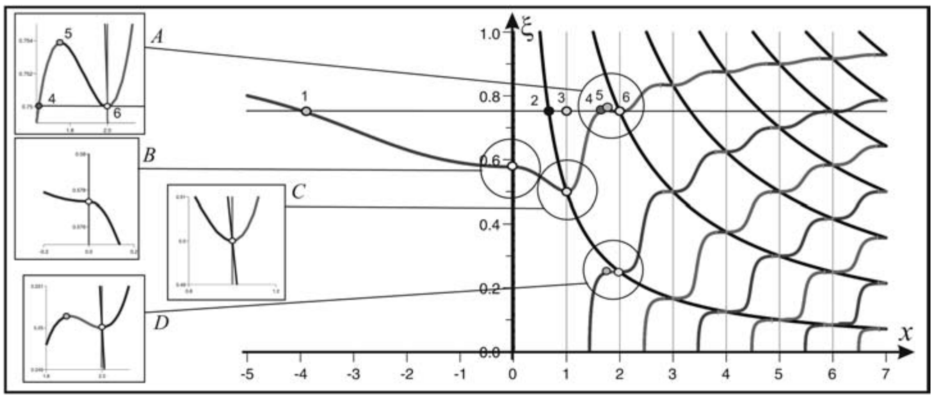


Fig. 1. Critical points of the characteristic function in Case 3.

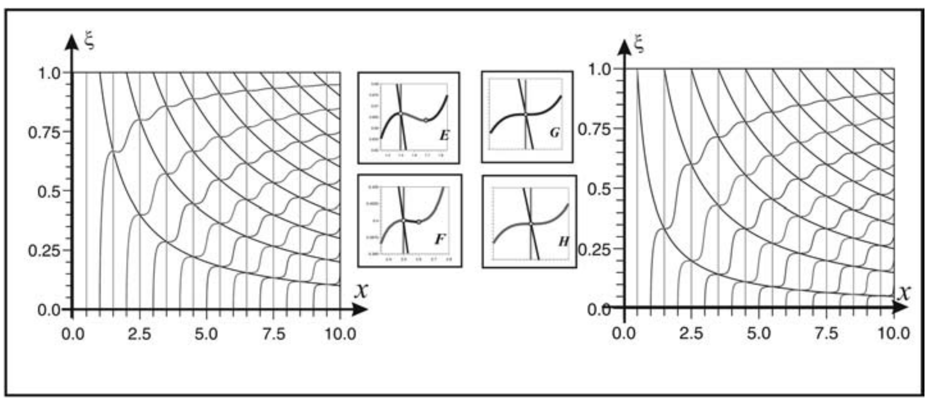


Fig. 2. Critical points of the characteristic function in Case 1, 5′ (the left figure, E and F) and Case 2, 4′ (the right figure, G and H).

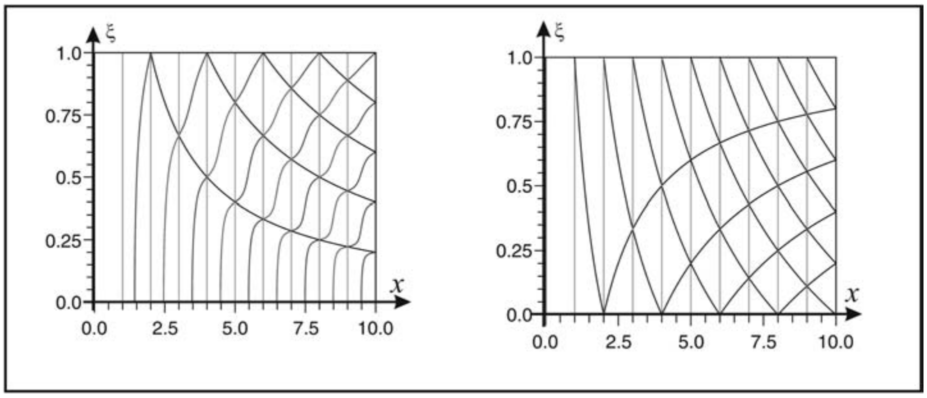


Fig. 3. Critical points of the characteristic function in Case 5 (the left figure) and Case 6′ (the right figure).

on the left side of the pole when value of parameter $\xi$ grows. This point joins with maximum point (see, Fig. 1, point 5) by particular value of parameter $\xi$ and we have another turning type bifurcation point. This point is bifurcation point of the critical points. Qualitative view of critical points curves is shown in figures A, B, C, D (see, Fig. 1).

Now we present few interesting results for critical points and constant eigenvalue points of the problems (1)–(3) and (1), (4), (3).

PROPOSITION 1. Critical points curves, the lines of zeroes and the hyperbolas of the poles points intersect in the constant eigenvalue points.

**Remark 4.1.** In Case 5 hyperbolas are curves of second order poles. In case of constant eigenvalue points they degenerate into the first order poles.

**Remark 4.2.** There are no critical points in Case 6 and Case 6′. In this case we have two families of the first order poles.

PROPOSITION 2. The curves of the first order poles in Case 6 and Case 6′ intersect only with zeroes lines and these cross points coincide with constant eigenvalue points.

PROPOSITION 3. In the case of nonlocal two-point boundary conditions (Cases 2 and 4) the type (maximum or minimum) of critical points does not change when we go through constant eigenvalue point (see, Fig. 2, G and H), while in case of boundary conditions 1, 3, 1′ and 3′ the type of critical points changes (see, Fig. 1, A and D; Fig. 2, E and F). In case of the nonlocal integral boundary condition the type of points changes (Case 5, see, Fig. 3) or there are no such points (Case 6 and Case 6′, see, Fig. 3).

PROPOSITION 4. In case of nonlocal two-point Case 1 and Case 3 boundary conditions bifurcation point of critical values is next to constant eigenvalue point (in Case

1 on the right, in Case 3 on the left), in case of two-point Case 2 and Case 4 boundary conditions and in Case 5 of integral nonlocal boundary condition there are no bifurcation points.

**Remark 4.3.** There is only one bifurcation point (which is constant eigenvalue point too) for $(x, \xi) = (\pi, 0.5)$ in Case 3 (see, Fig. 1, C).

**Remark 4.4.** In case of two-point nonlocal boundary conditions the curves of critical points converge to line $\xi = 1$ asymptotically, while in Case 5 of nonlocal integral boundary condition they reach this line in constant eigenvalues points. Curves of critical points reach line $\xi = 0$ in all the cases except one curve in Case 3.

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REZIUMĖ

***S. Pečiulytė, A. Štikonas. Kritinių ir kitų taškų pasiskirstymas kraštiniuose uždaviniuose su nelokaliosiomis kraštinėmis sąlygomis***

Šiame straipsnyje nagrinėjamas Šturmo ir Liuvilio uždavinys su viena nelokaliąja dvitaške arba integraline kraštine sąlyga. Analizuojami charakteristinės funkcijos kritiniai taškai, jų priklausomybė nuo nelokaliosios sąlygos parametrų.